Comparison of Oracles*

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Abstract

We analyze incomplete-information games where an oracle publicly shares information with privately informed players. One oracle dominates another if, in every game, it can match the set of equilibrium outcomes induced by the latter. Distinct characterizations of dominance and equivalence (mutual dominance) are provided for deterministic and stochastic signaling functions. The analysis highlights the role of common-knowledge components and develops a theory of information loops, thereby extending the seminal work of Blackwell (1951) to strategic environments and Aumann (1976)'s theory of common knowledge.

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1 Introduction

The motivation for this paper came from the seemingly distant domain of monetary policy, and specifically from public debates surrounding a Federal Open Market Committee press release. Small changes in wording are typically interpreted by market participants as signals about future policy and macroeconomic conditions (i.e., forward guidance). This raises a natural question: What do we know about a partially informed agent who provides noisy signals to strategic players (or markets) that themselves possess private and partial information? Similar questions arise from forecasters more generally: weather, sports, geopolitics, ratings agencies, or prediction markets, all release noisy signals to partially informed agents engaged in strategic interaction.

This paper examines incomplete-information games where players are partially informed, both privately and publicly, about the realized state. The private information is provided to every player by their specific partition, and the public information is disclosed by an external source, namely, an *oracle*.

Formally, an oracle holds a partition of the state space and communicates through a *signaling function* that is measurable with respect to the oracle's partition. Any such signaling scheme is a Blackwell experiment (see Blackwell, 1951), so an oracle can be viewed as *a generator of experiments* compatible with its information. Each experiment, combined with the players' private partitions, induces a "guided game" with its own set of equilibria. Notably, the oracle need not know what is common knowledge among the players.

The primary objective is to compare two oracles in terms of their ability to induce equilibria across all games. Oracle 1 is said to *dominate* Oracle 2 if, for every game and every signaling function available to Oracle 2, Oracle 1 has a signaling scheme that induces exactly the same set of equilibrium distributions over state-action profile pairs.

Throughout the paper we keep the players' private information fixed and vary only the oracle's signaling strategies. This stands in contrast to alternative notions that must hold *for every* possible configuration of players' private information. We also adopt a stringent notion of dominance: Oracle 1 dominates Oracle 2 if it can reproduce exactly the set of equilibrium outcome distributions that Oracle 2 can generate in any game. This makes the comparison independent of any particular equilibrium-selection rule or objective function of the oracle. A broader discussion of these modeling assumptions is given in Section 3.

The analysis distinguishes between deterministic and stochastic signaling strategies (of the oracles). A deterministic strategy maps the oracle's information set to a deterministic signal, whereas a stochastic strategy generates signals according to a probability distribution. For deterministic strategies, we characterize dominance via a *jointly more informative* (JMI) condition: Oracle 1 dominates Oracle 2 if and only if, for every signaling function of Oracle 2, Oracle 1 can match the players' joint posterior beliefs once their private information is taken into account (Theorem 1). Notably, JMI is a weaker notion than partition refinement, which is implied by Blackwell's notion of dominance, but mutual dominance under JMI forces the two oracle partitions to coincide (Theorem 2).

When oracles are allowed to employ stochastic signaling functions, the resulting posterior beliefs become more intricate, so that dominance requires additional criteria that hinge on two key elements derived from the players' information structures. The first element is the *common knowledge component* (CKC)—the minimal (inclusion-wise) set that all players commonly agree upon (see Aumann, 1976). Building on the structure of CKCs, we introduce a second essential concept: the *information loop*. To formally define information loops, we first partition the state space into disjoint CKCs. An information loop is then described as a closed path through the state space that links different CKCs via elements of an oracle's partition. See Figure 1 for an illustration.

Assuming that an oracle does not generate information loops, which includes the case where the entire state space comprises a unique CKC, we prove that it dominates the other oracle if and only if its partition refines that of the other in every CKC (Theorems 3 and 4). Importantly, the refinement condition does not follow from the JMI criterion (Example 1).

However, if a loop exists, the characterization becomes more complex, because information loops impose constraints on the information the oracle can convey. A loop consists of separate entry and exit points in every CKC, and these *pairs of states* go to the heart of Bayesian inference. Given a CKC, Bayesian updating depends on the ratio of signal-probabilities for the different states. Thus, a loop imposes constraints over such ratios, so the oracle is not free to signal any information it wants in one CKC, without restricting its ability to convey different information in another CKC. This is also a crucial aspect in the characterization of implementable joint posterior beliefs in Lagziel and Lehrer (2025).

The concept of information loops hints at a significant connection to Aumann's theory of common knowledge (Aumann, 1976). These loops introduce *an additional layer beyond common knowledge*, as they arise only through interactions with an external agent who is not part of the common-knowledge structure. These loops appear to be central to understanding how shared and differing information structures impact equilibrium outcomes in incomplete-information games. Section 7 therefore develops a detailed analysis of their foundational properties.

Our analysis identifies three essential properties of information loops: (i) *irreducibility*, (ii) *non-informative*, and (iii) *covering*. We establish necessary conditions for dominance in the general case (Theorem 5), relying on *irreducibility*, which requires that a loop cannot be

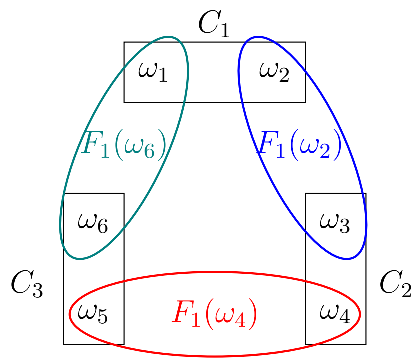

Figure 1: An illustration of a fully-informative and irreducible loop, which intersects three CKCs C_1, C_2 and C_3 with two states in each. F_1 denotes the partition of Oracle 1, and its elements intersect different CKCs.

reduced to a strictly smaller one (see Figure 1). The remaining two properties highlight how Oracle 2 may be constrained by the same information limitations that a loop imposes on Oracle 1. This may occur either because Oracle 2 possesses a corresponding loop that *covers* it or because Oracle 2 is *non–informative* on that loop and therefore provides no additional information conditional on its states. We prove that non-informative loops do not obstruct dominance (Theorem 6), and in the case of two CKCs, we characterize dominance through the covering property together with refinement in each CKC (Proposition 3).

Though the case of informative loops with more than two CKCs remains open, the resulting understanding of the aforementioned properties underpins another main result of the paper: a full characterization of *equivalent* oracles, defined through mutual dominance (Theorem 7).

1.1 Motivation and applications

Although Section 2 develops a complete motivating example, it is useful to discuss how our model applies more broadly, starting from the opening example in Kamenica and Gentzkow (2011). Recall that a fully informed prosecutor seeks to persuade an uninformed judge to convict a defendant. To achieve this, the prosecutor commits to a signaling structure designed to induce conviction with some probability.

Taking this example a step further, in many judicial systems the prosecutor must persuade not only a judge but also a *jury*.[1] Moreover, the prosecutor is typically *not* fully informed, while neither the judge nor the jury is completely uninformed. Each holds personal intuitions, prior beliefs, and subjective perspectives. In practice, the prosecutor faces informational constraints analogous to those encountered by the oracle in our framework.

[1] For example, the Sixth Amendment to the United States Constitution guarantees criminal defendants a speedy and public trial by an impartial jury.

In this analogy, the prosecutor is the oracle, the judge and jury are the players, and their private intuitions and priors correspond to the players' private information. Our partial order ranks prosecutorial information systems (e.g., evidence-disclosure regimes) by the sets of equilibrium verdicts they can induce across all such games. The same logic applies to the problem of belief polarization in our Section 2 example: different "oracles" (fact checkers) can be ranked by the equilibrium depolarization patterns they can implement.

This perspective highlights why the partial ordering of oracles is a central comparative notion in such environments: standard tools (e.g., concavification) may fail under partial and asymmetric information. In a similar set-up, Lagziel and Lehrer (2025) show that the feasible set of joint posterior beliefs is typically non-convex given a non-binary state space, when both the oracle and the players possess private and partial information. Consequently, a partial ordering of "experiment generators" becomes a powerful methodological tool for analyzing these problems. Practically, this means that in settings with partially informed judges/juries or central banks/markets, one can compare alternative information systems (transparency regimes, disclosure formats, types of forecasts) by asking which oracle dominates another, without solving a fully-fledged information-design problem.

1.2 Relation to literature

Blackwell (1951, 1953) provides the classic comparison of experiments for a single decision maker: one information structure dominates another if it yields weakly higher expected utility in every decision problem. We extend this framework along two dimensions. First, we move from decision problems to incomplete-information games, where the objects being compared are equilibria of "guided games" induced by an oracle's signals. Second, instead of a fixed experiment, an oracle is a generator of experiments compatible with its partition. Our dominance notion is therefore about the ability of one oracle to replicate the equilibrium outcome distributions that another oracle can induce, rather than optimizing a particular decision maker's payoff.

Brooks et al. (2024) strengthen Blackwell by requiring robustness to arbitrary auxiliary signals and decision problems and characterize *strong Blackwell dominance* between two signals. Their analysis compares two information sources (signals) that are robust to any external information source and decision problem. In contrast, we fix the players' private information structures and the prior, and compare oracles that can implement any experiment measurable with respect to their partitions. Our dominance relation is specific to a given configuration of players' information and to strategic interaction, not to a universal set of decision problems.

A related literature compares information structures in games and establishes partial orders. Peski (2008) analyzed zero-sum games, offering an analogous result to Blackwell's by characterizing when one information structure is more advantageous for the maximizer. Lehrer et al. (2010, 2013) analyze signaling and mediation in common-interest games and show how variants of Blackwell garbling characterize outcome equivalence. Likewise, Bergemann and Morris (2016) characterize dominance among two information structures through the concept of individual sufficiency, an extension of Blackwell's notion of garbling to n-player games. A common feature of these papers is that they compare *fixed* information structures (typically private signals) and use versions of Blackwell's garbling to capture dominance. We instead hold players' private information fixed and compare oracles that provide *public signals*, subject to an oracle-measurability constraint (the oracle cannot condition on players' private signals). For this reason, oracle dominance is not meant to coincide with, or simply specialize, Bergemann and Morris (2016) to public signals: even under a "public-signal-only" restriction, their order is a pairwise comparison of two fixed information structures, whereas our is a comparison over the menus of feasible public experiments generated by different oracle partitions. Dominance in our setting is driven by the implementable joint posterior beliefs, constrained by CKCs and information loops.

Another strand studies mediators in incomplete-information games who correlate players' actions through private recommendations, often without adding information about the realized state; see Forges (1993) and Gossner (2000) among others. In these models, the mediator's role is to coordinate actions so as to implement correlated equilibria. Our oracles differ in two respects: they provide *public signals* rather than private recommendations, and their role is not to coordinate actions but to change beliefs. As a result, dominance in our setting cannot be reduced to the richness of the correlated-equilibrium correspondence.

Our departure from existing dominance notions (see, e.g., Gossner, 2000 and Bergemann and Morris, 2016) lies in restricting information provision to *public disclosure*, given players' private information and the Oracle's information. This restriction imposes global measurability and consistency constraints that are absent in other frameworks. If players' private information is trivial, if the Oracle is allowed to issue private recommendations, or if the Oracle is agnostic about players' information and the comparison ranges over all possible information structures, these measurability constraints vanish, and our notion of dominance collapses to standard refinement or Blackwell-type comparisons. The conceptual contribution of this paper is to isolate and clarify the role of public disclosure in shaping the interaction between fixed players' private information and the Oracle's information, and to show how this interaction fundamentally alters the set of attainable equilibrium outcomes.

Our use of common knowledge components is rooted in the epistemic foundations of games. Aumann (1976) defines common knowledge and the induced partition into common knowledge components. Monderer and Samet (1989), Mertens and Zamir (1985), and Brandenburger and Dekel (1993) clarify how hierarchies of beliefs and type spaces encode such information. Our model builds on these studies by fixing the partition structures while varying only the oracle's public experiment. The novel constraints we study arise from *global* measurability across CKCs (via loops), not from additional complexity in private belief hierarchies. Information loops then formalize how public measurability links different CKCs and constrains the set of posterior profiles that an oracle can generate.

1.3 The structure of the paper

The paper is organized as follows. In Section 2, we provide a simple example to illustrate the key concepts of the paper. Section 3 presents the model and key definitions. Section 4 analyzes deterministic oracles, including a characterization of dominance and a proof that two-sided dominance implies the oracles are identical (given a unique CKC). In Section 5, we examine stochastic oracles and characterize dominance in the case of a unique CKC. Section 6 provides a characterization of dominance when there are no loops. Section 7 studies the properties of information loops. Section 8 outlines necessary and sufficient conditions for dominance, as well as a characterization of dominance given two CKCs. Finally, in Section 9 we characterize the equivalence relation between oracles. Section 10 concludes.

2 Fact checking and belief depolarization: an example

The problem of polarization, whether affective, ideological, or identity-driven, has received substantial attention in both public discourse and academic research (e.g., Arieli et al., 2021; Arieli et al., 2024, and Ikan et al., 2025). Technological progress has not mitigated this problem. Selective exposure to information and the ease of coordinated disinformation may instead amplify disagreement, making fact checking and depolarization particularly challenging. The following stylized example illustrates how our framework captures these forces.

Consider two individuals (or populations) who receive information from different sources about an unknown state $\omega \in \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with common prior $\mu = (\frac{1}{16}, \frac{4}{16}, \frac{5}{16}, \frac{6}{16})$. Their private information is represented by the partitions $\Pi_1 = \{\{\omega_1, \omega_3, \omega_4\}, \{\omega_2\}\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}\}$. To abstract from actions and payoffs, polarization is measured by the expected ℓ^1-disagreement $D(\tau) = \mathbb{E}[\|\mu^1_\tau - \mu^2_\tau\|_1]$, where μ^i_τ denotes player i's posterior given her private information and the public signal generated by τ.[2] If no public information is provided, that is, if the public signal is constant, each player conditions only on her private information. A direct computation yields $D(\tau^{\text{Const}}) = \frac{407}{480} \approx 0.848$.

Now consider a fact checker, referred to as Oracle 1, who seeks to reduce disagreement. Oracle 1 observes the public partition $F_1 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$. While a perfectly informed oracle could eliminate disagreement, in practice verifiable evidence is often limited. In this example, F_1 represents the informational constraint faced by the fact checker. If Oracle 1 fully reveals the realized F_1-cell, then $D(\tau_1^{\text{Full}}) = \frac{97}{120} \approx 0.808$, so revealing F_1 reduces disagreement relative to having no oracle.

Next, define an F_1-measurable experiment with two public signals s_1, s_2 by

$$\tau_1^{\text{Noisy}}(s_1 \mid \omega) = \begin{cases} 0, & \text{if } \omega \in \{\omega_1, \omega_3\}, \\ \frac{3}{4}, & \text{if } \omega \in \{\omega_2, \omega_4\}, \end{cases} \qquad \tau_1^{\text{Noisy}}(s_2 \mid \omega) = 1 - \tau_1^{\text{Noisy}}(s_1 \mid \omega).$$

This F_1-measurable experiment is a garbling of τ_1^{Full} and hence Blackwell-inferior to full revelation of F_1.[3] Nevertheless, a direct computation gives $D(\tau_1^{\text{Noisy}}) = \frac{31}{40} = 0.775 < D(\tau_1^{\text{Full}})$, so full revelation is strictly dominated for the purpose of minimizing expected disagreement.

Consider instead Oracle 2, endowed with the public partition $F_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. Notice that F_2 neither refines F_1 nor is refined by it, so neither oracle Blackwell-dominates the other. If Oracle 2 fully reveals the realized F_2-cell, then for every $\omega \in \Omega$, we get $\Pi_1(\omega) \cap F_2(\omega) = \Pi_2(\omega) \cap F_2(\omega)$. Thus the players condition on the same event at every state and therefore hold identical posterior beliefs, implying $D(\tau_2^{\text{Full}}) = 0$. In summary,

$$D(\tau^{\text{Const}}) > D(\tau_1^{\text{Full}}) > D(\tau_1^{\text{Noisy}}) > D(\tau_2^{\text{Full}}) = 0.$$

This example highlights several themes that recur throughout our analysis. First, players' private information substantially complicates the analysis, and the problem of polarization more generally, to the extent that even garbling need not be "inferior" to full revelation. Second, the oracle's objective in this example, and more generally in any strategic environ-

[2]The $D(\tau)$ is essentially the "β-polarization" in Arieli et al., 2024, when $\beta = 1$. The objective $D(\tau)$ is used only as a convenient summary statistic for posterior disagreement. The example is intended to illustrate informational feasibility and dominance comparisons, rather than to formulate an optimization problem for the oracle. One may microfound this objective via a coordination game in which equilibrium payoffs depend on the players' joint posterior beliefs, so that less disagreement is beneficial.

[3]Let τ_1^{Full} reveal the realized F_1-atom, with signals t for $\{\omega_1, \omega_3\}$ and f for $\{\omega_2, \omega_4\}$. Define a kernel K by $K(s_1 \mid t) = 0$, $K(s_2 \mid t) = 1$, and $K(s_1 \mid f) = \frac{3}{4}$, $K(s_2 \mid f) = \frac{1}{4}$. Then $\tau_1^{\text{Noisy}}(s \mid \omega) = \sum_{r \in \{t,f\}} K(s \mid r)\, \tau_1^{\text{Full}}(r \mid \omega)$ for all ω, so τ_1^{Noisy} is a garbling of τ_1^{Full}.

ment we consider, is to control the distribution of joint posterior beliefs, which encodes the entire hierarchy of beliefs. This objective is distinct from, and typically more restrictive than, coordination via correlated equilibrium in the sense of Forges (1993). Third, this example also motivates the general question: given players' partitions (possibly with multiple CKCs), when does Oracle 1 dominate Oracle 2 in the sense that for every F_2-measurable experiment τ_2 there exists an F_1-measurable experiment τ_1 inducing the same joint distribution of posterior beliefs (same polarization)?

3 The model

A *guided game* comprises a Bayesian game and an *oracle*. The oracle's role is to provide information that enables a different, and preferably broader, range of equilibria. It does so through signaling, and our analysis seeks to characterize the extent to which oracles can expand the set of equilibrium payoffs.

We begin by defining the underlying Bayesian game. Let $N = \{1, 2, \ldots, n\}$ be a finite set of $n \geq 2$ players, and let Ω denote a non-empty, finite state space. Each player $i \in N$ has a non-empty set of actions A_i and a partition Π_i over Ω, representing the information available to player i. Denote the set of action profiles by $A = \times_{i \in N} A_i$. The utility function for each player $i \in N$ is $u_i : \Omega \times A \to \mathbb{R}$, which maps states and action profiles to real-valued payoffs.[4]

To extend the basic game into a guided game, we introduce an oracle who provides public information before players choose their actions. The oracle is endowed with a partition F of the state space Ω, and a countable set S of possible signals.

A *signaling strategy* of the oracle is an F-measurable function $\tau : \Omega \to \Delta(S)$, where $\Delta(S)$ is the set of probability distributions on S with finite support. This function specifies the distribution over signals conditional on the realized state and must be measurable with respect to F. We denote by $\tau(s|\omega)$ the probability that the oracle sends signal s when the realized state is ω. A *deterministic signaling strategy* corresponds to a function $\tau : F \to S$ and hence to sub-partitions of F. We will refer to it as such when appropriate.

The guided game evolves as follows. First, the oracle publicly announces a strategy τ. Then, a state $\omega \in \Omega$ is drawn according to a common prior $\mu \in \Delta(\Omega)$ with full support. Each player i is privately informed of $\Pi_i(\omega)$, which is a set of states containing ω and also

[4]The underlying state space Ω represents payoff-relevant fundamentals only, referred to in Maschler et al. (2013) as the states of nature. Agents' information is modeled by their information partitions, and payoffs do not depend directly on agents' information or beliefs. The states of nature together with these partitions, when they are common knowledge among all players, determine the players' types, which are characterized by the full hierarchy of beliefs, à la Harsanyi.

an atom of player i's private partition. Finally, the signal $\tau(\omega) \in S$ is publicly announced in case τ is deterministic, or $s \in S$ is drawn according to $\tau(\omega)$ and is publicly announced in case τ is stochastic.

Let the join[5] $\Pi_i \vee F'$ denote the updated information (i.e., partition) of player i given Π_i and some partition F'. In case τ is a deterministic function, let $\mu^i_{\tau|\omega} = \mu(\cdot|[\Pi_i \vee \tau](\omega)) \in \Delta(\Omega)$ denote player i's posterior belief after observing $\Pi_i(\omega)$ and $\tau(\omega)$. In case τ is stochastic, let $\mu^i_{\tau|\omega,s} = \mu(\cdot|\Pi_i(\omega), \tau, s) \in \Delta(\Omega)$ denote player i's posterior belief after observing $\Pi_i(\omega)$ and a realized signal s according to $\tau(\omega)$, and let $\mu_{\tau,s} = \left\{(\mu^i_{\tau|\omega,s})_{i\in N} : \omega \in \Omega \text{ s.t. } \tau(s|\omega) > 0\right\}$ be the set of *joint posteriors* associated with τ and a signal s, across all relevant states. The joint posteriors capture each player's belief about the realized state and their beliefs about others' beliefs, as well as higher-order beliefs. We use μ_τ to denote the distribution over all joint posteriors induced by τ across all signals, and use $\text{Post}(\tau) = \text{Supp}(\mu_\tau)$ to denote its support. Thus, every strategy τ yields an incomplete-information game $G(\tau) = (N, (A_i)_{i\in N}, \mu_\tau, (u_i)_{i\in N})$. When there is no risk of ambiguity, we denote the incomplete-information game without τ by G.

3.1 Partial ordering of oracles

To discuss the role of the oracle in the current framework, one needs a relevant solution concept. Thus, let us define the following notion of a Guided equilibrium, which incorporates the oracle's strategy. Formally, let $\sigma_i : \Pi_i \times S \to \Delta(A_i)$ be a strategy of player i. A tuple $(\tau, \sigma_1, \ldots, \sigma_n)$ is a *Guided equilibrium* if $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium in the incomplete-information game $G(\tau)$.

The notion of a Guided equilibrium defines a partial ordering of oracles, i.e., a partial relation over their partitions according to the sets of equilibria. To define this relation, let $\text{NED(G}(\tau)) \subseteq \Delta(\Omega \times \text{A})$ be the set of distributions over $\Omega \times A$ induced by Nash equilibria given G and τ.[6] Now consider two oracles, Oracle 1 and Oracle 2, and denote the generic partition and strategy of Oracle j by F_j and τ_j, respectively. Using these notations we define a partial ordering of oracles as follows.

Definition 1 (Partial ordering of oracles). *Fix the players' information structures. We say that* Oracle 1 dominates Oracle 2, *denoted* $F_1 \succeq_{\text{NE}} F_2$, *if for every* τ_2 *and game* G, *there exists* τ_1 *such that* $\text{NED(G}(\tau_1)) = \text{NED(G}(\tau_2))$.

[5]Coarsest common refinement of Π_i and F'; following the definition of Aumann (1976).

[6]Note that a Nash equilibrium $(\sigma^*_i, ..., \sigma^*_n)$ induces a probability distribution over $\Omega \times A$. Specifically, fix ω and an action profile a, the probability of (ω, a) under the equilibrium strategy $(\sigma^*_i, ..., \sigma^*_n)$ and the signaling function τ is given by $\mu(\omega) \sum_{s\in S} \tau(s|\omega) \prod_{i=1}^n \sigma^*_i(a_i|\Pi_i(\omega), s)$. Since multiple equilibria can exist, $\text{NED(G}(\tau))$ is a subset of $\Delta(\Omega \times A)$.

Intuitively, Oracle 1 dominates Oracle 2 if, whatever information policy Oracle 2 uses and whatever game is played, Oracle 1 can choose a policy that yields exactly the same set of equilibrium outcome distributions. Note that a direct comparison of the games' equilibria is problematic because the players' strategies depend on the oracles' signaling functions.

Three points are worth noting here. First, one could consider defining dominance between oracles in a more robust manner by allowing players' information structures to vary over a set of possible partitions. If dominance were required to hold uniformly over all such partitions, the comparison problem would become substantially simpler. In particular, the set of admissible information structures would include the case of trivial private information. In that case, the common-knowledge structure admits no loops and, by our results, oracle dominance reduces to partition refinement. The substantive challenge in our framework arises precisely because players' information partitions are fixed and predetermined, so that dominance must be assessed under binding measurability and common-knowledge constraints.

Second, Definition 1 compares oracles via equality of equilibrium distributions rather than inclusion. Analogously, one can define dominance in terms of equilibrium payoff sets, either via equality or inclusion. Outcome–based dominance implies the corresponding payoff–based notions, but the former is more natural when the objective is to control actions and aggregate outcomes, whereas payoff–based notions suit environments that focus on agents' utilities. We adopt this stronger notion to avoid imposing a specific equilibrium-selection rule (which may diverge from the Pareto frontier), and to keep the dominance relation independent of any objective functions that might be assigned to the oracles in a parallel setup.

Third, another way to compare oracles is to treat them as strategic players in a sender–receiver game, assign payoff functions, and say that Oracle 1 is more informative than Oracle 2 if it obtains (weakly) higher equilibrium payoffs in every such game. This approach faces several difficulties: (i) equilibria are typically multiple (so oracle payoffs depend on an arbitrary selection rule); (ii) it presumes that the oracle knows the players' private information partitions (and more generally their type structure); and (iii) it ties the comparison of information structures to particular oracle objectives. By contrast, our dominance notion compares oracles solely through the equilibrium outcome distributions they can induce for a *fixed* configuration of players' information, assumes only that these information structures are common knowledge among the players (not necessarily the oracle), and thus allows us to analyze the oracles' information partitions and the associated notions independently of any objectives they might have.

4 Partial ordering of deterministic oracles

Our first main result characterizes the notion of dominance among oracles, assuming they are restricted to deterministic strategies. That is, throughout this section, we only consider oracles that use deterministic functions, namely $\tau_j : F_j \to S$ for every oracle j, and we can relate to every such strategy as a partition.

The characterization is based on the ability of one oracle to *match* the players' joint posterior beliefs, for any given strategy of the other oracle. More formally, we say that Oracle 1 is *jointly more informative* than Oracle 2, if for every strategy τ_2, there exists a strategy τ_1 that simultaneously matches the posterior partition of every player i.

Definition 2. Oracle 1 is jointly more informative (JMI) than Oracle 2*, if for every deterministic* τ_2*, there exists a deterministic* τ_1 *such that* $\Pi_i \vee \tau_1 = \Pi_i \vee \tau_2$ *for every player* i*.*

In other words, one oracle is jointly more informative than another if it can always ensure that every player has the same information as provided by the other oracle, taking into account the players' private information as well as the publicly available signal. Another way to express the same relation is via partition refinements, as given in the following observation.

Observation 1. Oracle 1 *is jointly more informative than* Oracle 2 *if and only if for every* $F_2' \subseteq F_2$,[7] *there exists* $F_1' \subseteq F_1$ *such that* $\Pi_i \vee F_1' = \Pi_i \vee F_2'$*, for every player* i*.*

Observation 1 is simply a partition-based restatement of Definition 2, since every F_i-measurable deterministic strategy induces a sub-partition of F_i and vice versa. The next example shows that JMI may differ from the standard refinement order.

The partial ordering generated by the notion of "jointly more informative than" need not coincide with the notion of "finer than". For example, consider the trivial case in which the players have perfect information. Then, every oracle is JMI than the other, independently of their partitions. Nevertheless, in Section 4.2.1 we prove that, given a unique CKC, if Oracle 1 is JMI than Oracle 2 and vice versa, then their partitions coincide.

One can also bridge the gap between the notions of JMI and refinement by considering the possibility that the players' partitions are not fixed.[8] If we allow players' partitions to vary over all possible partitions, including the trivial one, then JMI implies refinement:

[7] A partition F_2' is a subset of partition F_2 if the σ-field generated by F_2' is a subset of the σ-field generated by F_2.

[8] This resembles the condition of strong Blackwell dominance, in the context of decision problems, in Brooks et al. (2024).

Oracle 1 must be able to match any deterministic strategy of Oracle 2, so F_1 refines F_2 (at least weakly).

4.1 First characterization result - deterministic oracles

The first main result of this section shows that, when oracles use deterministic strategies, dominance is equivalent to being jointly more informative. The proof is constructive: starting from a violation of JMI, we build a game in which some equilibrium distribution induced by Oracle 2 cannot be replicated by any deterministic strategy of Oracle 1. (Unless stated otherwise, all proofs are deferred to the Appendix.)

Theorem 1. *Assume that oracles are deterministic. Then,* Oracle 1 *dominates* Oracle 2 *if and only if* Oracle 1 *is jointly more informative than* Oracle 2*.*

Intuitively, the "if" direction is immediate: if Oracle 1 can replicate the information of Oracle 2 for every deterministic strategy, the induced incomplete-information games have the same joint posteriors and hence the same equilibrium outcome distributions. For the reverse direction, the proof constructs a game, based on proper scoring rules employing the Kullback–Leibler divergence, with a unique equilibrium which maps to the players' joint posterior belief. Thus, any failure of JMI for Oracle 1 relative to Oracle 2 generates an equilibrium distribution that Oracle 1 cannot mimic.[9]

Remark 1. *We repeatedly use the implication that, since μ is fixed, any difference in equilibrium expected payoffs when following τ_1 instead of τ_2 translates into a difference in the induced distributions over $\Delta(\Omega \times A)$. The reverse deduction, however, is not necessarily true, as different such distributions may yield the same expected payoffs.*

4.2 Common knowledge components

Theorem 1 characterizes dominance using the notion of JMI, which does not necessarily imply refinement of partitions. This raises the general question of whether the notion of JMI leads, in some way, to a refinement of partitions while taking into account the players' private information.

To study this aspect in the context of games, rather than in decision problems, we first need to define the notion of a "Common Knowledge Component." Following Aumann (1976), an event E is a *common knowledge component* (CKC) of the partitions $\Pi_1, \Pi_2, \ldots, \Pi_n$ if it is an element in the meet $\bigwedge_{i=1}^n \Pi_i$, which is the finest common coarsening of all the partitions,

[9] In the extended version of the paper, we construct a finite game where failure to meet the JMI condition leads to different equilibrium payoffs.

That is, a CKC is a minimal event (inclusion wise) that is common knowledge among the players. Within each CKC, all posteriors and hence equilibrium payoffs are determined solely by the information available inside that component, so players' expected payoffs and the oracle's impact can be analyzed CKC-by-CKC.

Using this definition, we can now debate the general hypothesis of whether an JMI oracle also has a finer partition in every CKC. The answer for this question is no. The following example shows that even in the case of a unique CKC, the fact that Oracle 1 is JMI than Oracle 2 does not imply that F_1 refines F_2.

Example 1. JMI *does not imply refinement in every* CKC*, and refinement in every* CKC *does not imply* JMI*.*

To see that JMI does not imply refinement in every CKC, consider the information structure $\Pi_1 = \{\{\omega_1, \omega_4\}, \{\omega_2\}, \{\omega_3\}\}$, $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}\}$, and $\Pi_3 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$. One can see that there exists a unique CKC, as the finest common coarsening of all players' partitions is Ω. The oracles, however, have the following partitions: $F_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ and $F_2 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$.

Both oracles can either withhold all information or fully disclose their information, thereby ensuring that all players become fully informed of the realized state. In fact, these are all the possible signaling functions of Oracle 2. On the other hand, Oracle 1 can also signal the partition $F_1' = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}$, which provides complete information to players 1 and 2 but provides no information to player 3. Thus, Oracle 1 is JMI than Oracle 2, while neither of the two partitions is finer than the other.

Another aspect of this example, which resonates with the key insight of the stochastic setting in Section 5, is that there exists a stochastic strategy τ_2 that Oracle 1 cannot imitate. Specifically, consider the stochastic strategy τ_2 given in Figure 2. One can verify that there exists no τ_1 that yields the same vectors of posteriors as the stated strategy τ_2, and this hinges on the fact that F_1 does not refine F_2.

The key issue is that in the deterministic case each state is associated with a *unique* public signal, so JMI guarantees coincidence of the entire profile of posteriors and hence of the induced Bayesian game. Under stochastic signaling, however, each state can generate multiple signals with *different weights*, so the same partitions can induce different joint posteriors. This richer structure is not fully captured by players' interim partitions (i.e., given any deterministic information conveyed by the oracles), and creates both within-CKC and across-CKC difficulties that require stronger conditions than JMI.

To demonstrate that refinement in every CKC does not imply JMI, consider the following example with two players whose partitions are $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_4, \omega_5\}, \{\omega_3, \omega_6\}\}$ and $\Pi_2 =$

$\tau_2(s\|\omega)$	s_1	s_2
ω_1	1/3	2/3
ω_2	2/3	1/3
ω_3	1/3	2/3
ω_4	2/3	1/3

Figure 2: A stochastic F_2-measurable strategy of Oracle 2.

$\{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$. In this case, there are two CKCs, $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4, \omega_5, \omega_6\}$. Next, assume the two oracles have the following partitions, $F_1 = \{\{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_5, \omega_6\}\}$, $F_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$, as illustrated in Figure 3. Observe that in every CKC, F_1 refines F_2.

Now consider a completely revealing, deterministic strategy τ_2 that maps the three different partition elements of F_2 to three different signals: $\tau_2(s_1|\omega_1) = \tau_2(s_1|\omega_2) = 1$, $\tau_2(s_2|\omega_3) = \tau_2(s_2|\omega_4) = 1$, and $\tau_2(s_3|\omega_5) = \tau_2(s_3|\omega_6) = 1$. Can Oracle 1 produce a signaling function τ_1 such that $\Pi_i \vee \tau_1 = \Pi_i \vee \tau_2$ for every player i?

Note that under τ_2, neither player can distinguish ω_1 from ω_2. Therefore, in order for τ_1 to satisfy $\Pi_i \vee \tau_1 = \Pi_i \vee \tau_2$ for every i, the strategy τ_1 must map all F_1 partition elements to the same signal. Consequently, under τ_1, Player 1 cannot distinguish ω_4 from ω_5, which is achievable given τ_2. We therefore conclude that Oracle 1 is not JMI than Oracle 2, even though F_1 refines F_2 in every CKC. However, in the special case where Ω consists of a single CKC, refinement does imply JMI.

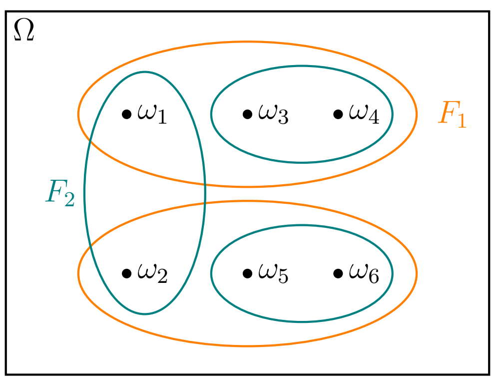

Figure 3: Refinement in every CKC does not imply JMI. Suppose $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_4, \omega_5\}, \{\omega_3, \omega_6\}\}$ and $\Pi_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}\}$. There are two CKCs, $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4, \omega_5, \omega_6\}$. Consider F_1 (orange) and F_2 (teal) depicted in the figure. Despite F_1 refines F_2 in every CKC, F_1 is not jointly more informative than F_2.

4.2.1 Two-sided JMI implies equivalence in every CKC

Though we substantiated that an JMI oracle need not have a finer partition in every CKC, this does hold in case *both* oracles dominate one another, under deterministic signaling strategies. The following theorem provides this equivalence by stating that, given a specific CKC, both oracles dominate each other if and only if their partitions coincide.

Theorem 2. *Fix a unique* CKC*. Then, Oracle i is* JMI *than Oracle $-i$ for every i if and only if $F_1 = F_2$.*

By applying the result within each CKC, the theorem asserts that the partitions F_1 and F_2 are equivalent in every CKC if and only if they are mutually JMI within that CKC, given any *fixed* set of players' partitions. As a result, the issue of CKCs arises naturally in the context of deterministic oracles and becomes even more significant when studying stochastic ones, as examined in Section 5.

5 Partial ordering of (stochastic) oracles

This section extends the dominance analysis to general (stochastic) signaling strategies. Our main result characterizes dominance under a single CKC by relating it to an inclusion condition on the joint posteriors induced by the oracles' strategies.

5.1 A unique CKC

In this section, we characterize oracle dominance under the assumption that Ω consists of a unique CKC. Specifically, we prove in Theorem 3 that, given a unique CKC, Oracle 1 dominates Oracle 2 if and only if F_1 refines F_2. This is also equivalent to the condition that for every strategy τ_2, there exists a strategy τ_1 such that the inclusion condition holds (by itself *and as an equality*), and it is also equivalent to the condition that the set of distributions over posteriors profiles are identical (namely, that for every strategy τ_2, there exists a strategy τ_1 such that $\mu_{\tau_1} = \mu_{\tau_2}$). While this result has significant merits on its own, it also serves as a foundational building block for subsequent results.

Theorem 3. *Assume that Ω comprises a* unique *common knowledge component. Then, the following are equivalent:*

(i) F_1 *refines* F_2*;*

(ii) $F_1 \succeq_{\mathrm{NE}} F_2$*;*

(iii) For every τ_2, there exists τ_1, so that $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$*;*

(iv) For every τ_2, there exists τ_1, so that $\text{Post}(\tau_1) = \text{Post}(\tau_2)$*;*

(v) For every τ_2, there exists τ_1, so that $\mu_{\tau_1} = \mu_{\tau_2}$.

Theorem 3 presents an intriguing *equivalence* between partition refinements and the inclusion condition. Notably, this result applies to any information structure with a unique CKC, independent of any specific game. Furthermore, the refinement condition implies that Oracle 1 can effectively mimic any strategy of Oracle 2, allowing Oracle 1 to support the same sets of distributions on $\Omega \times A$ induced by Nash equilibria in incomplete-information games for any given τ_2. The equivalence between $F_1 \succeq_{\text{NE}} F_2$ and the last condition in Theorem 3 extends beyond the case of a unique CKC. However, as demonstrated in Section 5.2, Theorem 3 as a whole generally fails when there are multiple CKCs.

5.2 More than one CKC: two examples

The partition-refinement condition in Theorem 3 ensures that Oracle 1 can produce the *exact* same strategy as Oracle 2. This however, hinges on the existence of a unique CKC. In case there are several CKCs, Oracle 1 may need to follow a different strategy in order to match the distribution on posteriors generated by τ_2. Namely, τ_1 may require more signals than τ_2, even if both oracles have the same (complete) information in every CKC. Let us provide a concrete example for this.

Example 2. *A mimicking strategy τ_1 may require more signals than τ_2.*

Consider a uniformly distributed state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with two players whose private information is $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}\}$. The oracles have the following partitions $F_1 = \{\{\omega_1, \omega_3\}, \{\omega_2\}, \{\omega_4\}\}$ and $F_2 = \{\{\omega_1\}, \{\omega_3\}, \{\omega_2, \omega_4\}\}$. Notice that there are two CKCs, $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$, and both oracles have complete information in each of these components. That is, F_1 refines F_2 in every CKC, and vice versa.

Consider the stochastic strategy τ_2 given in Figure 4. Notice it is F_2-measurable, as $\tau_2(s|\omega_2) = \tau_2(s|\omega_4)$ for every signal s, but not F_1-measurable.

The set of τ_2-posteriors $\text{Post}(\tau_2)$ is

$$\text{Post}(\tau_2) = \left\{ \begin{array}{ll} (e_i, e_i), & \forall\ 1 \le i \le 4, \\ \left(\left(\frac{3}{7}, \frac{4}{7}, 0, 0\right), e_j\right), & j = 1, 2, \\ \left(e_k, \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)\right), & k = 3, 4 \end{array} \right\},$$

$\tau_2(s\|\omega)$	s_1	s_2	s_3
ω_1	0	1/2	1/2
ω_2	1/3	2/3	0
ω_3	0	2/3	1/3
ω_4	1/3	2/3	0

Figure 4: A stochastic F_2-measurable strategy of Oracle 2.

where the i^{th} coordinate of e_i is 1, and we can now try to mimic τ_2 using an F_1-measurable strategy. First, this requires at least two signals to distinguish between ω_1 and ω_2, as well as ω_3 and ω_4. Second, the posterior $\left(\left(\frac{3}{7}, \frac{4}{7}, 0, 0\right), e_1\right)$ requires another signal s so that $\tau(s|\omega_1) = \alpha > 0$ and $\tau(s|\omega_3) = \frac{4}{3}\alpha > 0$. However, the F_1-measurability requirement implies that $\tau(s|\omega_3) = \alpha$, and the τ_2-posterior $\left(e_3, (0, 0, \frac{1}{2}, \frac{1}{2})\right)$ necessitates that $\tau(s|\omega_4) = \alpha$ as well. These conditions are given in Table (a) within Figure 5.

$\tau_1(s\|\omega)$	s_3	s_4	s_5
ω_1	α	β	0
ω_2	$\frac{4}{3}\alpha$	0	γ
ω_3	α	β	0
ω_4	α	0	γ

(a)

$\tau_1(s\|\omega)$	s_3	s_4	s_5	s_6
ω_1	1/2	1/3	0	1/6
ω_2	2/3	0	1/3	0
ω_3	1/2	1/3	0	1/6
ω_4	1/2	0	1/3	1/6

(b)

Figure 5: A strategy τ_1, either with 3 signals as given in Table (a), or with 4 signals as in Table (b).

Evidently, it must be that $\alpha, \beta, \gamma > 0$ in order to mimic τ_2, but the second and fourth rows in Table (a) cannot jointly sum to 1 unless $\alpha = 0$, which eliminates the possibility of a well-defined mimicking strategy. Thus, in order to mimic the stated strategy τ_2, Oracle 1 requires an additional signal as presented in Table (b), in Figure 5. To conclude, though the oracles' partitions refine one another in every CKC, they cannot always produce the exact same strategy when trying to mimic each other.

Remark 2. *Note that dominance does not imply refinement in general. To see this, consider the information structure* $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, F_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}$ *and* $F_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$*. For every strategy* τ_2*, one can devise a strategy* τ_1 *that yields the same distribution over posterior beliefs. Evidently,* Oracle 2 *provides the players with no additional information regarding states* ω_1 *and* ω_2*, and this allows* Oracle 1 *to replicate* τ_2 *on* ω_3 *and* ω_4*, accordingly.*

6 Multiple CKCs and no loops

We now turn to the general setting in which the players' information structures induce any (finite) number of CKCs. Assume that $C_1, \ldots, C_l$ are mutually exclusive CKCs such that $\Omega = \bigcup_{j=1}^{l} C_j$. A key aspect of our analysis is the presence of measurability constraints, where different CKCs are connected by atoms of the oracles' partitions, forming what we call an (information) *loop*. Generally, a loop is an ordered sequence of states from different CKCs such that the partition of an oracle groups together distinct pairs of states from different CKCs, creating a closed path.

Definition 3. *An F_i-loop is a sequence $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, where $m + 1 \equiv 1$ and $m \geq 2$, such that*

- *$\omega_j, \overline{\omega}_j \in C_{r_j}$ and $\omega_j \neq \overline{\omega}_j$ for all $j = 1, \ldots, m$.*[10]
- *$\omega_{j+1} \in F_i(\overline{\omega}_j)$ for all $j = 1, \ldots, m$.*
- *$C_{r_j} \neq C_{r_{j+1}}$ for all $j = 1, \ldots, m$.*
- *The sets $\{\overline{\omega}_j, \omega_{j+1}\}$ are pairwise disjoint for all $j = 1, \ldots, m$.*

To understand information loops, one can view the CKCs as the vertices of a graph. An edge connects two CKCs if there exist ω_{j+1} and $\overline{\omega}_j$ such that they belong to the same F_i-partition element (this corresponds to the second requirement). An information loop then parallels an Eulerian graph, where there is a walk that includes every edge exactly once (the last requirement in the definition) and ends back at the initial vertex (hence the requirement $m+1 \equiv 1$). As noted at the beginning of this section, the key aspect of the general analysis is to consider the case when the oracle partition atoms intersect different CKCs, so we require that $C_{r_j} \neq C_{r_{j+1}}$ for all $j = 1, \ldots, m$.

An example of an F_1-loop is provided in Figure 6.(a), which depicts a loop consisting of six states across three CKCs. Note that a loop can intersect the same CKC multiple times, as long as the sets $\{\overline{\omega}_j, \omega_{j+1}\}$ remain pairwise disjoint for each j.[11]

We use the concept of a loop in Theorem 4. This theorem builds on the assumption that F_1 contains no loops and extends Theorem 3 by showing that one oracle dominates another if the former's partition refines that of the latter in every CKC.

Theorem 4. *Assume there is no F_1-loop. Then,* Oracle 1 *dominates* Oracle 2 *if and only if F_1 refines F_2 in every* CKC.

[10] Here C_{r_j} refers to the CKC that contains the j-th pair of states $(\omega_j, \overline{\omega}_j)$.
[11] A loop intersects a given CKC once if there is a unique pair of states $(\omega_j, \overline{\omega}_j)$ from the loop that lies in that CKC.

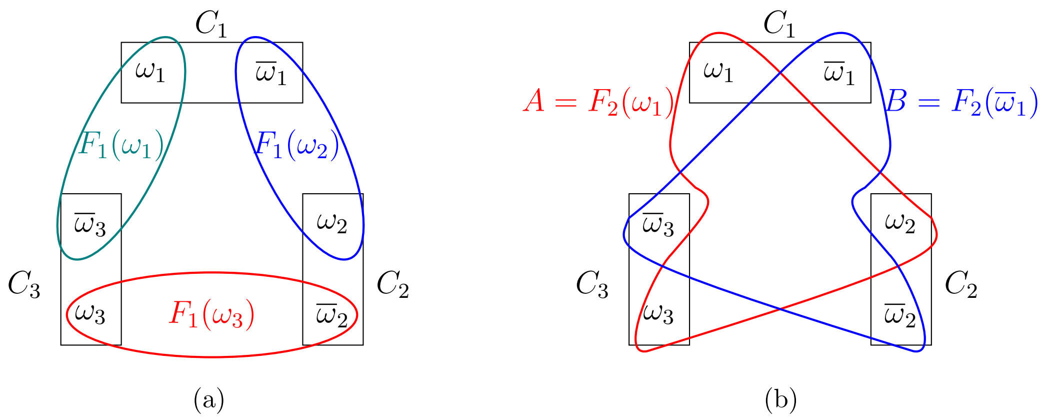

Figure 6: Figure (a) depicts an F_1-loop with three CKCs and six states overall. Figure (b) illustrates how the F_1-loop, presented in (a), is non-balanced with respect to F_2. Namely, F_2 has two elements $A = \{\omega_1, \omega_2, \omega_3\}$, and $B = \{\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\}$ such that the number of transitions from A to B are 3, while the reverse equals 0.

The "only if" direction is straightforward: consider a CKC C and every game supported on C, the desired result follows from Theorem 3. For the "if" direction, we construct a graph whose vertices are CKCs and whose edges connect CKCs that are intersected by a common atom of F_1. The no-loop assumption guarantees that each connected component of this graph is a tree. On each such tree, we start from a root CKC, where Theorem 3 is used to match μ_{τ_2} by an F_1-measurable strategy. We then extend τ_1 inductively along the tree. At each step, F_1-measurability pins down the behavior on the boundary atom connecting two CKCs, while refinement within the new CKC and the ability to split signals ensure that μ_{τ_2} can still be matched there. Iterating this procedure over all CKCs yields a global F_1-measurable strategy τ_1 with $\mu_{\tau_1} = \mu_{\tau_2}$, establishing that $F_1 \succeq_{\text{NE}} F_2$.

7 Information loops

Previous sections have examined the problem of oracle dominance in the absence of loops, considering either a unique CKC or multiple CKCs without loops. However, in order to confront the general question of dominance in the presence of information loops, we need to have a clear understanding of their properties and implications.

Specifically, when an F_1-loop exists, it may create challenges for Oracle 1 in mimicking Oracle 2, because loops introduce measurability constraints across CKCs. Although Oracle 1 can mimic Oracle 2 within each CKC, it may be impossible to do so simultaneously across CKCs if the required combined strategy is not measurable with respect to F_1. This suggests that any F_1-loop must satisfy certain conditions to ensure that such a strategy is indeed F_1-measurable. The first condition that we study, which turns out to be a necessary condition for dominance, is generally referred to as F_2-*balanced*.

The idea starts with an F_1-loop. We examine all states in this loop and determine how they can be covered by two F_2-measurable sets. In other words, the loop is divided into two disjoint sets, each contained in an F_2-measurable set, denoted A and B. Next, we count the number of transitions along the loop from A to B, where the entry point into one CKC is through a state in A and the exit is through a state in B. We do the same for transitions from B to A. An F_1-loop is called F_2-*balanced* if the number of transitions between A and B is equal in both directions. The formal definition follows.

Definition 4. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_{-i}-*balanced *if for every F_{-i}-measurable partition of the loop's states into two disjoint sets $\{A, B\}$ such that $\cup_j\{\omega_j, \overline{\omega}_j\} \subseteq A \cup B$, it follows that:*

$$\#(A \to B) := |\{j; \omega_j \in A \text{ and } \overline{\omega}_j \in B\}| = |\{j; \omega_j \in B \text{ and } \overline{\omega}_j \in A\}| =: \#(B \to A). \quad (1)$$

Note that an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, where $\omega_j \in F_2(\overline{\omega}_j)$ for all $j = 1, \ldots, m$, is F_2-*balanced.* Figure 6.(b) shows a partition of the F_1-loop in 6.(a) into two F_2-measurable sets A and B. Since $\#(A \to B) = 3$ while $\#(B \to A) = 0$, the F_1-loop fails to be F_2-balanced.

Why are balanced loops crucial? Consider, for example, a non-balanced loop as depicted in Figure 6, and assume that $\tau_2(s|\omega) = \frac{1}{2} - \frac{1}{4}\mathbf{1}_{\{\omega \in A\}}$ for some signal $s \in S$. This imposes a specific 1 : 2 ratio between any two states described in each CKC, so that $\Pi_i \frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)} = \frac{1}{8}$. However, since $\overline{\omega}_i$ and ω_{i+1} belong to the same F_1 partition element, the measurability constraints on Oracle 1 along the loop require that $\tau_1(s|\overline{\omega}_i) = \tau_1(s|\omega_{i+1})$, hence $\Pi_i \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = 1$ for any s in the support of all states. In other words, Oracle 1 cannot match the ratio dictated by τ_2, therefore the key proportionality Lemma 1 (in Appendix A.3) does not hold in at least one CKC.

If the loop were balanced—say, with $A = \{\overline{\omega}_1, \omega_2\}$ and $B = \{\omega_1, \overline{\omega}_2, \omega_3, \overline{\omega}_3\}$—then the same strategy τ_2 would yield $\Pi_i \frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)} = 1$, as required. In general, when all loops are balanced, this discrepancy is eliminated for any two such sets A and B. The notion of balanced loops is closely related to the following notion of *covered loops*, which implies that an F_1-loop can be decomposed to loops of F_2.

Definition 5. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_{-i}-*covered *if*

- *The set $\{1, ..., m\}$ is partitioned to disjoint sets of indices, $J, I_1, ..., I_r$, i.e., $\{1, ..., m\} = J \cup (\cup_{t=1}^r I_t)$.*

- *For each* $t = 1, ..., r$, $\Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I_t}$ *is an* F_{-i}*-loop, also referred to as a* sub-loop.[12]
- $J = \{j; \omega_j \in F_{-i}(\overline{\omega}_j)\}$.

The cover is order-preserving *if every* F_{-i}*-loop* $\Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I_t}$ *in the cover follows the same ordering of pairs as the* F_i*-loop.*

In simple terms, the definition states that, given an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, we can partition its states into several F_2-loops and a set of states where $\omega_j \in F_2(\overline{\omega}_j)$. Figure 7 (a) depicts an F_1-loop consisting of $((\omega_j, \overline{\omega}_j))_{j=1,\ldots,4}$, which is covered by two F_2-loops: $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3)$ and $(\omega_2, \overline{\omega}_2, \omega_4, \overline{\omega}_4)$. In this case, the set J (defined in Definition 5) is empty. Figure 7 (b) depicts a case in which $J = \{2, 4\}$, and $(\omega_1, \overline{\omega}_1, \overline{\omega}_3, \omega_3)$ forms an F_2-loop, yet it is not an F_2-sub-loop of the original F_1-loop since $\overline{\omega}_1$ is linked to $\overline{\omega}_3$ instead of ω_3. Actually, if we set $A = \{\omega_2, \overline{\omega}_2, \omega_4, \overline{\omega}_4, \omega_1, \omega_3\}$ and $B = \{\overline{\omega}_1, \overline{\omega}_3\}$, which are F_2-measurable, then $\#(A \to B) = 2$, but $\#(B \to A) = 0$, so the F_1-loop is not F_2-balanced. Finally, note that the sub-loops in Figure 7 (a) are order-preserving. By contrast, the sub-loop $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3, \omega_2, \overline{\omega}_2)$ in Figure 7 (c) does not preserve the ordering of the pairs as the F_1-loop, since the pair $(\omega_3, \overline{\omega}_3)$ appears before $(\omega_2, \overline{\omega}_2)$. In Section 8, we show that order-preservation is needed to obtain a necessary condition for oracle dominance.

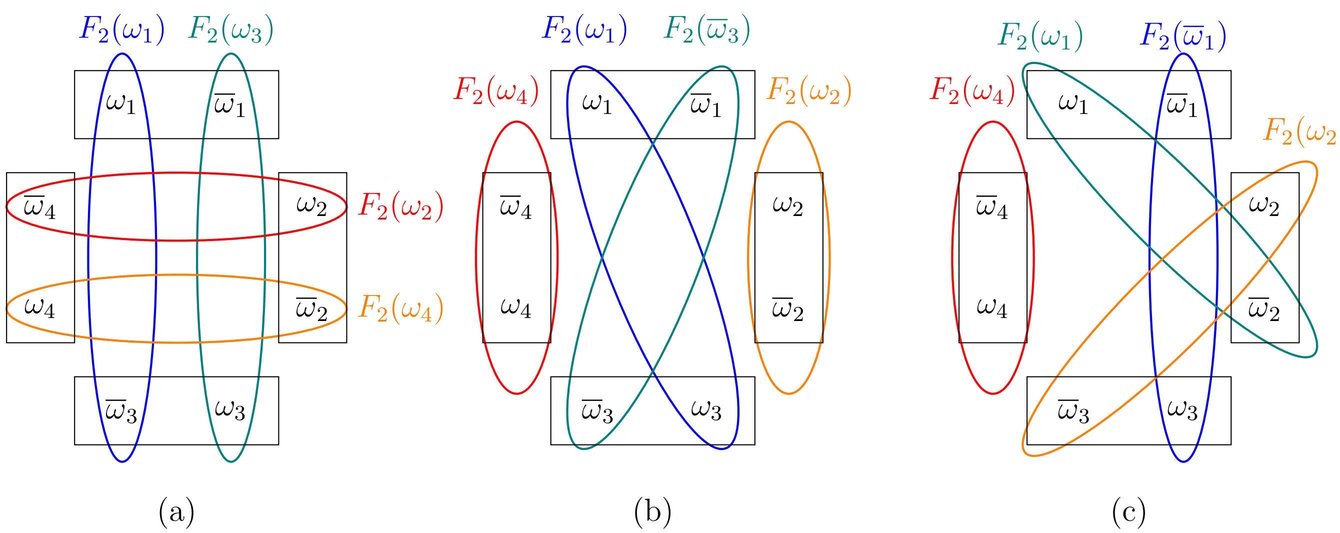

Figure 7: Two states connected by a colored line are in the same information set of F_2. In (a), the F_2-sub-loops that cover the F_1-loop are order-preserving, i.e., following the ordering of pairs in the original F_1-loop, whereas the sub-loop in (c) is not order-preserving. (b) illustrates a case where $(\omega_1, \overline{\omega}_1, \overline{\omega}_3, \omega_3)$ forms an F_2 loop, but it is not an F_2-sub-loop of the original F_1-loop.

[12] The order of the pairs $(\omega_j, \overline{\omega}_j)$ in the F_{-i}-loop does not have to coincide with their order under the F_i-loop. For instance, an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \omega_3, \overline{\omega}_3)$ might be covered by the following F_2-loop $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3, \omega_2, \overline{\omega}_2)$.

The following Proposition 1 proves that an F_1-loop is F_2-balanced if and only if it is F_2-covered. This proposition assists with the proof of Theorem 5 below, which provides a necessary condition for dominance.

Proposition 1. *Let* $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ *be an* F_1*-loop. The following statements are equivalent:*

i. *The loop is* F_2*-balanced;*

ii. *The loop is* F_2*-covered;*

iii. *For every* F_2*-measurable function* $f : \{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m\} \to (0, \infty)$,

$$\prod_{i=1}^{m} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1.$$

The next two properties that we study are *irreducible* and *informative* loops. Starting with the former, an F_i-loop is irreducible if it does not have a *sub-loop*, namely, there exists no 'smaller' F_i-loop that comprises a strictly smaller set of states taken solely from the original loop. Our analysis would use irreducible loops as building blocks to decompose and compare loops generated by the oracles' partitions.

Definition 6. *Let* $L_i = (\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ *be an* F_i*-loop. We say that the loop is* irreducible *if there exists no strict subset of the set* $\{\omega_j, \overline{\omega}_j : j = 1, \ldots, m\}$ *that forms an* F_i*-loop.*

We use the definition of an irreducible loop in the context of covers as well, stating that a cover is *irreducible* if every loop in the cover is irreducible. Furthermore, the idea of irreducible loops is closely related to the concept of covers, and specifically to the set $J = \{j; \omega_j \in F_{-i}(\overline{\omega}_j)\}$ given in Definition 5 above. Specifically, if there exists an F_i-loop with a pair of states $(\omega_j, \overline{\omega}_j)$ such that $\overline{\omega}_j \in F_i(\omega_j)$, then it cannot be irreducible unless it comprises only 4 states.[13] We typically refer to such cases where $\overline{\omega}_j \in F_i(\omega_j)$ as *non-informative* because Oracle i cannot distinguish between the two states. This condition is essentially equivalent to every F_1-loop being F_2-balanced at 0, meaning that for any choice of the specified F_2-measurable sets A and B, the number of transitions between these sets is zero. The following Definition 7 formalizes the notions of *non-informative* and *fully-informative* loops. The non-informative case will be used in Theorem 6 as a sufficient condition for dominance.

[13] In general, the smallest possible loop has at least 4 states, so any such loop is, by definition, irreducible.

Definition 7. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_k-*non-informative *if $F_k(\omega_j) = F_k(\overline{\omega}_j)$ for every j. The loop is F_k-*fully-informative *(F_k-*informative*) if $F_k(\omega_j) \neq F_k(\overline{\omega}_j)$ for every j (for some j, respectively).*

To understand the motivation behind this definition, consider any F_1-loop denoted by $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$. If this loop is F_2-non-informative, it suggests that the ratios $\frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)}$ equals 1 for every signal s supported on these states. In simple terms, conditional on any $\{\omega_i, \overline{\omega}_i\}$, Oracle 2 does not provide any additional information, so the constraints that an F_1-loop imposes on Oracle 1 in every CKC (i.e., that the product of probability ratios along the loop equals 1) are met by the measurability requirements of F_2.

The following proposition summarizes key properties of informative and irreducible loops. It states that an irreducible loop intersects every CKC at most once and must be fully informative (unless it has only 4 states). In addition, the proposition shows that an informative loop has a fully-informative sub-loop, as well.

Proposition 2. *Consider an F_i-loop L_i.*

- *If L_i intersects the same* CKC *more than once, then it is not irreducible.*
- *If L_i is irreducible and consists of at least* 6 *states, then it is F_i-fully-informative.*
- *If L_i is F_i-informative, then it has an F_i-fully-informative sub-loop.*
- *If L_i is F_i-fully-informative, then it can be decomposed to irreducible F_i-loops.*
- *If L_i is not irreducible, then either it intersects the same* CKC *more than once, or it has at least 4 states in the same partition element of F_i.*

We use this proposition in the following section to provide necessary and sufficient conditions for the dominance of one oracle over another.

8 Necessary and sufficient conditions for dominance

In this section, we address the general case where F_1 has loops, which imposes constraints on Oracle 1 *across* CKCs. Due to the complexity of this problem, we divide our analysis into two parts: a necessary condition for dominance presented in Theorem 5, and a sufficient condition given in Theorem 6. These theorems depend strongly on the properties of information loops, and specifically on the notions of covers, irreducibility and non-informative.

Starting with the necessary conditions, the following theorem, which builds on Propositions 1 and 2, states that if Oracle 1 dominates Oracle 2, then besides the refinement condition in every CKC, already established in Theorem 4, it must be that every F_1-loop is covered by loops of F_2. In addition, it states that every irreducible F_2-loop that covers an irreducible F_1-loop is order-preserving, essentially stating that the two loops coincide.

Theorem 5. *If* Oracle 1 *dominates* Oracle 2*, then:*

- *F_1 refines F_2 in every* CKC*;*
- *Any F_1-loop has a cover by F_2-loops; and*
- *Fix an irreducible F_1-loop L_1. If the F_2-cover has a single sub-loop, which is also irreducible, then the cover is order-preserving.*

The proof of the first part follows directly from Theorem 4. The proof of the second part relies on Proposition 1 by assuming that an F_1-loop is not F_2-balanced, and constructing a strategy τ_2 that Oracle 1 cannot mimic without violating measurability constraints. The last part relies on Proposition 2, as well as on Lemma 1 in Appendix A.3, by depicting a two-signal strategy τ_2 that one cannot mimic without following the same order of pairs throughout the F_2-loop.

Next, we use the understanding regarding covered and balanced loops to present a sufficient condition for dominance, which indirectly requires that any loop is balanced at 0—meaning that there are no transitions between sets A and B. This leads to the following Theorem 6, which uses the non-informative notion for dominance.

Theorem 6. *If F_1 refines F_2 in every* CKC *and every F_1-loop is F_2-non-informative, then* Oracle 1 *dominates* Oracle 2*.*

Though we do not yet provide a full characterization, it becomes rather clear that the requirement that every F_1-loop is F_2-balanced should be the main focus, as it is a necessary condition, as well as a sufficient one when the balance is set to zero.

In what follows, we show that for two CKCs the balance condition is both necessary and sufficient. This case is especially tractable: our earlier results reduce the problem to checking a finite set of loop constraints, which we can verify directly.

To build intuition, consider the scenario with two CKCs, where $C_i = \{\omega_i, \bar{\omega}_i\},\ i = 1, 2$, featuring an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ across the four states. Fix some τ_2 and assume the loop is F_2-balanced. There are then only two possibilities: either the loop is F_2-non-informative, or it is also an F_2-loop. The first possibility was covered in Theorem 6, while the second allows Oracle 1 to meet the constraints imposed by the F_1-loop when attempting to mimic τ_2.

Proposition 3. *Assume there are only two* CKCs. *Then,* Oracle 1 *dominates* Oracle 2 *if and only if F_1 refines F_2 in every* CKC *and any F_1-loop is F_2-balanced.*

9 Equivalent oracles

In this section we tackle a parallel question to dominance, which is the problem of oracles' equivalence. Specifically, we characterize necessary and sufficient conditions such that both oracles dominate one another simultaneously, as formally given in the following definition:

Definition 8. *F_1 is* equivalent *to F_2, denoted $F_1 \sim F_2$, if the two oracles dominate one another, that is, if $F_i \succeq_{\text{NE}} F_{-i}$ for every $i = 1, 2$.*

Based on the results for the case that loops do not exist and the case of two CKCs, equivalence between oracles obviously requires two-sided refinement within every CKC (i.e., equivalence), and that every F_i-loop is F_{-i}-balanced for every Oracle i. This, however, is insufficient and equivalence also requires that every irreducible F_i-loop with at least 6 states is also an irreducible F_{-i}-loop. This result is given in the following Theorem 7.

Theorem 7. *F_1 is* equivalent *to F_2 if and only if for every* Oracle *i, the partition F_i refines F_{-i} in every* CKC, *any F_i-loop has a cover of F_{-i}-loops, and every irreducible F_i-loop with at least* 6 *states is an irreducible F_{-i}-loop.*

The equivalence condition concerning irreducible loops is based on the ability of both oracles to follow similar measurability constraints when signaling to players in every CKC. That is, if one oracle is constrained by an information loop, then we require the other to follow suit. Yet, this still raises the question of why we need to focus on irreducible loops. To understand this, consider a single partition element of F_i that intersects at least two CKCs where each intersection contains at least two states. This evidently generates a non-informative loop, because all pairs are non-informative. But as long as the other oracle cannot distinguish between the two states in each pair, the ability to separate different pairs in different CKCs is not needed, as each pair is common knowledge among the players themselves within every CKC.

The proof of Theorem 7 also builds on an intermediate irreducibility notion that we refer to as *type-2 irreducible loop.* More formally, an F_i-loop is type-2 irreducible if it does not have four states from the same partition element of F_i. This notion refines that of fully-informative loops (as every type-2 irreducible loop is fully-informative), but also weakens that of irreducible loops, because a type-2 irreducible loop can intersect the same CKC multiple times, and so be decomposed into sub-loops.

The notion of type-2 irreducible loops is crucial for our analysis and results, but also in a more general manner. We use type-2 irreducible loops to generate the basic elements, *building blocks*, upon which two oracles must match one another (in terms of their information). These building blocks are referred to as *clusters* and they are constructed as follows. First, we take the set of type-2 irreducible loops. Then, we consider such loops that intersect the same CKC and consider them as connected. Next, we take the transitive-closure of this relation, which yields disjoint sets of connected type-2 irreducible loops. Finally, we take every such set (of connected loops) and consider all the CKCs that it intersects; this is a cluster. We prove that the oracles' partitions match one another in each of these clusters. That is, the clusters are the basic structure upon which we derive an equivalence, and later extend it to "simpler" connections between clusters that involve only a single partition element of F_i.

10 Conclusion

This paper develops a comparative theory of generators of public information in incomplete-information games. We study an external oracle endowed with a partition of the state space who provides public signals to players holding heterogeneous private information. Signals may be deterministic or stochastic but must be measurable with respect to the oracle's partition, so each signaling scheme induces a Blackwell experiment. Fixing players' information structures and the prior, we introduce a dominance relation over oracles based on their ability to replicate each other's sets of equilibrium outcome distributions across all games. This extends Blackwell (1951) from decision problems to incomplete-information games and from single experiments to generators of experiments. Oracle dominance further connects to Aumann's theory of common knowledge through the central roles of CKCs and information loops.

For deterministic oracles, we show that dominance is equivalent to being jointly more informative: Oracle 1 can match every player's posterior partition induced by any signaling scheme of Oracle 2 (Theorem 1). For stochastic oracles with a unique CKC, dominance reduces to partition refinement (Theorem 3). When players' information generates multiple CKCs but the dominating oracle has no loops, dominance is characterized by refinement within each CKC (Theorem 4).

Characterizing dominance in the presence of information loops is more subtle, since loops link distinct CKCs through measurability constraints and restrict the joint posterior profiles an oracle can induce. We analyze these constraints via balanced and covered loops, irreducible and fully informative loops, and non-informative loops. Dominance implies refinement within each CKC and that every loop of the dominating oracle be covered by loops of the dominated one, with irreducible loops matched in an order-preserving manner (Theorem 5). Refinement together with non-informative loops is sufficient (Theorem 6), and in the special case of two CKCs, Proposition 3 shows that dominance is fully characterized by refinement and loop balance. Building on these ideas, we characterize oracle equivalence (mutual dominance) via two-sided refinement, loop covers, and matched irreducible loops (Theorem 7).

Our results provide a framework for ranking public information systems by the sets of equilibrium distributions they can induce, without imposing an objective on the information provider or restricting equilibrium selection. The analysis shows that in strategic environments with private information, public information cannot be evaluated solely by local informativeness. Global measurability constraints across CKCs, captured by information loops, determine which joint beliefs and hence equilibrium outcomes are implementable.

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A Appendices

A.1 Proof of Theorem 1

Proof. One direction is immediate: if Oracle 1 is JMI than Oracle 2, then for every deterministic F_2-measurable τ_2 there exists a deterministic F_1-measurable τ_1 such that $\Pi_i \vee \tau_1 = \Pi_i \vee \tau_2$ for all i. Hence the induced Bayesian games $G(\tau_1)$ and $G(\tau_2)$ are isomorphic, so $F_1 \succeq_{\text{NE}} F_2$.

For the converse, assume dominance and fix τ_2. Define the following game G_{KL}, where each player i choose an action $p_i \in \Delta(\Omega)$, interpreted as a reported belief.[14] The payoff is $u_i(p_i, \omega) = \log p_i(\omega)$ for $\omega \in \Omega$, which is a proper scoring rule (see, e.g.,Good (1952) and Savage (1971), among others). As a proper scoring rule, the unique maximizer (and equilibrium strategy) is player i's truthful belief. Thus, for every τ_j, the set $\text{NED}(G_{\text{KL}}(\tau_j))$ is a singleton whose marginal on the space of joint posterior beliefs coincides with μ_{τ_j}. By dominance, there exists τ_1 such that $\text{NED}(G_{\text{KL}}(\tau_1)) = \text{NED}(G_{\text{KL}}(\tau_2))$ and we get $\mu_{\tau_1} = \mu_{\tau_2}$ as well as $\text{Post}(\tau_1) = \text{Post}(\tau_2)$.

Because τ_1 and τ_2 are deterministic, each joint posterior profile in $\text{Post}(\tau_j)$ is of the form $(\mu(\cdot \mid (\Pi_i \vee \tau_j)(\omega)))_{i \in N}$ for some state ω. Therefore, equality of the conditional distributions $\mu(\cdot \mid (\Pi_i \vee \tau_1)(\omega)) = \mu(\cdot \mid (\Pi_i \vee \tau_2)(\omega))$ implies $(\Pi_i \vee \tau_1)(\omega) = (\Pi_i \vee \tau_2)(\omega)$ for all i and ω. Thus, $\Pi_i \vee \tau_1 = \Pi_i \vee \tau_2$ for every player i, which completes the proof. □

A.2 Proof of Theorem 2

Proof. Fix a unique CKC. The "if" direction is trivial, so assume that F_i is JMI than F_{-i} for every $i = 1, 2$, and let us prove that $F_1 = F_2$. Assume, to the contrary, that $F_1 \neq F_2$. W.l.o.g, there exist $\omega_1 \neq \omega_2$, such that $F_1(\omega_1) = F_1(\omega_2)$ whereas $F_2(\omega_1) \neq F_2(\omega_2)$.

Consider the partition $F_2' = \{F_2(\omega_1), (F_2(\omega_1))^c\}$. By assumption, there exists a partition F_1' such that $\Pi_i \vee F_1' = \Pi_i \vee F_2'$, for every player i. Denote $A = F_1'(\omega_1) \cap F_2(\omega_1)$, $B = F_1'(\omega_1) \cap (F_2(\omega_1))^c$, $C = (F_1'(\omega_1))^c \cap (F_2(\omega_1))^c$, and $D = (F_1'(\omega_1))^c \cap F_2(\omega_1)$.

If there exists a player i such that $\Pi_i(\omega_1) = \Pi_i(\omega_2)$, then $\omega_2 \in (F_1' \vee \Pi_i)(\omega_1)$, while $\omega_2 \notin (F_2' \vee \Pi_i)(\omega_1)$, which contradicts the equation $\Pi_i \vee F_1' = \Pi_i \vee F_2'$. Thus, for every $(\omega, \omega') \in A \times B \cup A \times D \cup B \times C$ and for every player i, we conclude that $\Pi_i(\omega) \neq \Pi_i(\omega')$.

Because this is a unique CKC, every two states ω and ω' have a connected path, in the sense that there exists a finite sequence of states starting with ω and ending with ω' where every two adjacent states are in the same information set of some player. Fix such a path from ω_1 to ω_2, and denote it by $(\omega_1, \omega_{1.1}, \omega_{1.2}, \ldots, \omega_{1.l}, \omega_3, \ldots, \omega_2)$ where $\{\omega_{1.t} : 1 \leq t \leq l\} \in C$ and $\omega_3 \in D$. This holds, w.l.o.g., because states in A are directly connected (through a partition

[14] The "KL" stands for the *Kullback–Leibler divergence* as this scoring rule builds on relative entropy.

element of some player) only to states in $A \cup C$, and the same holds for states in B that are directly connected only to states in $B \cup D$. Note that $\omega_{1.t} \in (F_1(\omega_1))^c$ for every t and $\omega_3 \in F_2(\omega_1) \cap (F_1(\omega_1))^c$.

Now consider the partition $F_1'' = \{F_1(\omega_1), (F_1(\omega_1))^c\}$. By assumption, there exists a partition F_2'' such that $\Pi_i \vee F_1'' = \Pi_i \vee F_2''$, for every player i. Denote $A' = F_1(\omega_1) \cap F_2''(\omega_1)$, $B' = (F_1(\omega_1))^c \cap F_2''(\omega_1)$, $C' = (F_1(\omega_1))^c \cap (F_2''(\omega_1))^c$, and $D' = F_1(\omega_1) \cap (F_2''(\omega_1))^c$.

Similarly to the previous analysis, states in A' are directly connected only to states in $A' \cup C'$, and states in B' are directly connected only to states in $B' \cup D'$. In addition, note that $\omega_1 \in F_1(\omega_1) \cap F_2(\omega_1) \subseteq A'$, $\omega_{1.t} \in (F_1(\omega_1))^c \subseteq B' \cup C'$ for every t, and $\omega_3 \in F_2(\omega_1) \cap (F_1(\omega_1))^c \subseteq B'$. If $\omega_{1.1} \in B'$, we can make a direct connection between A' and B', which yields a contradiction. Thus, $\omega_{1.1} \in C'$, and the sequence $(\omega_{1.1}, \omega_{1.2}, \ldots, \omega_{1.l_1}, \omega_3)$ which starts in C' and ends in B' has at least one direct connection between B' and C'. A contradiction, as well. Thus, for every $\omega_1 \neq \omega_2$, we conclude that $F_1(\omega_1) = F_1(\omega_2)$ if and only if $F_2(\omega_1) = F_2(\omega_2)$, and the result follows. □

A.3 Proportionality Lemma

Lemma 1. *Fix two distinct signals $\{s_1, s_2\}$ and assume that the partition $F_2 = \{A_1, A_2, \ldots, A_m\}$ has m elements. Let $p_1, p_2, \ldots, p_m$ be m distinct probabilities such that all ratios of two distinct numbers from the set $\mathbb{A} = \{p_j, 1 - p_j : j = 1, 2, \ldots, m\}$ are pairwise different.*[15] *Define*

$$\tau_2(s_1|A_j) = 1 - \tau_2(s_2|A_j) = p_j, \quad \forall 1 \leq j \leq m. \tag{2}$$

Assume a unique CKC. *If* $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$, *then for every signal $t \in \text{Supp}(\tau_1)$ there exists a signal $s \in \{s_1, s_2\}$ and a constant $c > 0$ such that $\tau_1(t|\omega) = c\tau_2(s|\omega)$ for every $\omega \in \Omega$.*

Proof. Assume, to the contrary, there exists a signal $t \in \text{Supp}(\tau_1)$ such that for every signal $s_i \in \{s_1, s_2\}$, there exist two states $\omega_1, \omega^* \in \Omega$ such that

$$\frac{\tau_1(t|\omega_1)}{\tau_2(s_i|\omega_1)} \neq \frac{\tau_1(t|\omega^*)}{\tau_2(s_i|\omega^*)}. \tag{3}$$

Note that $\tau_2(s_i|\omega) > 0$ for every s_i and ω, so the fractions are well defined. In addition, it must be that either $\tau_1(t|\omega_1) > 0$ or $\tau_1(t|\omega^*) > 0$, so assume that $\tau_1(t|\omega_1) > 0$. Because ω_1 and ω^* are in the same CKC, there exists a finite sequence $(\omega_1, \omega_2, \omega_3, \ldots, \omega^*)$ such that every two adjacent states are in the same partition element for some player. Using the definition of τ_2, it follows that in every joint posterior $(\mu^l_{\tau_2|\omega,s_i})_{l \in N} \in \text{Post}(\tau_2)$, the coordinates relating to $\Pi_l(\omega)$

[15] To achieve this, one can consider m distinct prime numbers $r_1 < r_2 < \cdots < r_m$. Define $\mathbb{T}_0 = \mathbb{Q}$, and for every $j \geq 1$, let $\mathbb{T}_j$ be the extended field of $\mathbb{T}_{j-1}$ with $\sqrt{r_j}$. Take $p_j \in \mathbb{T}_j \setminus \mathbb{T}_{j-1}$.

are strictly positive (for every player l and every signal s_i). Because $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$, it follows that $\tau_1(t|\omega) > 0$ for every $\omega \in \{\omega_1, \omega_2, \ldots, \omega^*\}$, as well.

Using Bayes' rule and for every $\omega, \omega' \in \Pi_l(\omega'')$, we get

$$\frac{\mu^l_{\tau_2|\omega'',s_i}(\omega)}{\mu^l_{\tau_2|\omega'',s_i}(\omega')} = \frac{\tau_2(s_i|\omega)}{\tau_2(s_i|\omega')} \cdot \frac{\mu(\omega)}{\mu(\omega')}.$$

Note that $\frac{\tau_2(s_i|\omega)}{\tau_2(s_i|\omega')} = 1$ if and only if $F_2(\omega) = F_2(\omega')$, and otherwise, the ratio $\frac{\tau_2(s_i|\omega)}{\tau_2(s_i|\omega')}$ is given by $c \in \{\frac{x}{y} :\ x, y \in \mathbb{A}\}$. Thus, for every such s_i where $\mu^l_{\tau_2|\omega'',s_i}(\omega) \cdot \mu^l_{\tau_2|\omega'',s_i}(\omega') > 0$, there exists a unique $c \in \{\frac{x}{y} :\ x, y \in \mathbb{A}\} \cup \{1\}$ such that

$$\frac{\mu^l_{\tau_2|\omega'',s_i}(\omega)}{\mu(\omega)} = c \cdot \frac{\mu^l_{\tau_2|\omega'',s_i}(\omega')}{\mu(\omega')}.$$

In case $c = 1$, then the last equation holds for every signal s_i.

By the inclusion criterion, for every joint posterior $(\mu^l_{\tau_1|\omega_2,t})_{l\in N}$ generated by τ_1, there exists a joint posterior $(\mu^l_{\tau_2|\omega'',s_i})_{l\in N}$ generated by τ_2, such that the two are identical. We thus conclude that

$$\frac{\mu^{l_1}_{\tau_1|\omega_2,t}(\omega_1)}{\mu(\omega_1)} = \frac{\mu^{l_1}_{\tau_2|\omega'',s_i}(\omega_1)}{\mu(\omega_1)} = c_1 \cdot \frac{\mu^{l_1}_{\tau_2|\omega'',s_i}(\omega_2)}{\mu(\omega_2)} = c_1 \cdot \frac{\mu^{l_1}_{\tau_1|\omega_2,t}(\omega_2)}{\mu(\omega_2)},$$

and

$$\frac{\mu^{l_2}_{\tau_1|\omega_2,t}(\omega_2)}{\mu(\omega_2)} = \frac{\mu^{l_2}_{\tau_2|\omega'',s_i}(\omega_2)}{\mu(\omega_2)} = c_2 \cdot \frac{\mu^{l_2}_{\tau_2|\omega'',s_i}(\omega_3)}{\mu(\omega_3)} = c_2 \cdot \frac{\mu^{l_2}_{\tau_1|\omega_2,t}(\omega_3)}{\mu(\omega_3)},$$

as well. Using Bayes' rule, the last two equations are equivalent to

$$\begin{aligned} \tau_2(s_i|\omega_1) &= c_1 \cdot \tau_2(s_i|\omega_2) = c_1 \cdot c_2 \cdot \tau_2(s_i|\omega_3), \\ \tau_1(t|\omega_1) &= c_1 \cdot \tau_1(t|\omega_2) = c_1 \cdot c_2 \cdot \tau_1(t|\omega_3). \end{aligned}$$

These equations hold for every s_i in case $c_1 = c_2 = 1$, and otherwise hold for a specific signal, which could be taken as s_1 without loss of generality. One can continue inductively along the sequence $(\omega_1, \omega_2, \omega_3, \ldots, \omega^*)$ to get

$$\tau_2(s_i|\omega_1) = c_1 \cdot \tau_2(s_i|\omega_2) = \cdots = [\Pi_{k\geq 1} c_k] \cdot \tau_2(s_i|\omega^*), \tag{4}$$

$$\tau_1(t|\omega_1) = c_1 \cdot \tau_1(t|\omega_2) = \cdots = [\Pi_{k\geq 1} c_k] \cdot \tau_1(t|\omega^*). \tag{5}$$

Dividing Equation (5) by Equation (4), we get $\frac{\tau_1(t|\omega_1)}{\tau_2(s_i|\omega_1)} = \frac{\tau_1(t|\omega*)}{\tau_2(s_i|\omega*)}$, which contradicts (3). $\square$

A.4 Proof of Theorem 3

Proof. Note that $(v) \Rightarrow (iv) \Rightarrow (iii)$ is immediate, since equality of μ_τ implies equality of supports, which implies inclusion. Also, $(i) \Rightarrow (ii)$ is immediate by Oracle 1's ability to replicate every strategy of Oracle 2. Next, to see that $(ii) \Rightarrow (v)$, follow the same game G_{KL} used in the proof of Theorem 1.

Finally, we will prove that $(iii) \Rightarrow (i)$. If F_1 does not refine F_2, there exists ω_0 and ω^*, so that $F_1(\omega_0) = F_1(\omega^*)$ and $F_2(\omega_0) \neq F_2(\omega^*)$. Consider the signaling function τ_2 defined in Lemma 1 and take any strategy τ_1. Assume, to the contrary that $\mathrm{Post}(\tau_1) \subseteq \mathrm{Post}(\tau_2)$. According to Lemma 1, for every signal $t \in \mathrm{Supp}(\tau_1)$ there exists a signal $s_i \in \mathrm{Supp}(\tau_2)$ and a constant $c > 0$ such that $\tau_1(t|\omega) = c\tau_2(s_i|\omega)$ for every ω. In addition, the measurability condition of τ_1 imply that $\tau_1(t|\omega_0) = \tau_1(t|\omega^*)$ for every signal t. Thus, $\tau_2(s_i|\omega_0) = \tau_2(s_i|\omega^*)$ and this contradicts the definition of τ_2. This establishes the equivalence between all stated conditions. □

A.5 Proof of Theorem 4

Proof. The "only if" direction follows by restricting attention to a single CKC and applying Theorem 3.

For the converse, assume that $F_1|_C$ refines $F_2|_C$ for every CKC C and that F_1 has no loops. Let H be the graph whose vertices are all CKCs and where C–C' is an edge iff some atom of F_1 intersects both. The absence of F_1-loops implies every connected component of H is a tree. Fix one such tree T and root it at some CKC $C_0 \in T$.

Given an arbitrary F_2-measurable strategy τ_2, we construct an F_1-measurable τ_1 on $\bigcup_{C \in T} C$ by induction on distance from C_0. On C_0, since $F_1|_{C_0}$ refines $F_2|_{C_0}$ and C_0 is a unique CKC, Theorem 3 yields an $F_1|_{C_0}$-measurable strategy with $\mu_{\tau_1}|_{C_0} = \mu_{\tau_2}|_{C_0}$.

Now let C be at distance $d \geq 1$ with parent C'. The edge C–C' corresponds to a *unique* atom E of F_1 meeting both CKCs, so F_1-measurability fixes τ_1 on $E \cap C$ to match its (already defined) values on $E \cap C'$. On the remaining F_1-atoms inside C, choose an $F_1|_C$-measurable extension so that $\mu_{\tau_1}|_C = \mu_{\tau_2}|_C$. This is feasible because $F_1|_C$ refines $F_2|_C$, and any mismatch in the number of signals on $E \cap C$ can be handled by splitting signals without affecting $\mu_{\tau_1}|_{C'}$.

Proceeding over all CKCs in T (and then over all components of H) yields an F_1-measurable τ_1 with $\mu_{\tau_1} = \mu_{\tau_2}$. Therefore Oracle 1 dominates Oracle 2. □

A.6 Proof of Proposition 1

Proof. **iii** $\Rightarrow$ **i**. Suppose that $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is not F_2-balanced. It means that there is a partition $\{A, B\}$ s.t. $\#(A \to B) \neq \#(B \to A)$. Define $f(\omega) = \mathbf{1}_{\{\omega \in A\}} + 2 \cdot \mathbf{1}_{\{\omega \in B\}}$. We obtain,

$$\prod_{i=1}^{m} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = \left(\frac{1}{2}\right)^{\#(A \to B)} \cdot 2^{\#(B \to A)} \neq 1.$$

This contradicts **iii**.

i $\Rightarrow$ **ii**. Assume **i** . For every i, let $D_i = \{\omega_j; \omega_j \in F_2(\omega_i)\} \cup \{\overline{\omega}_j; \overline{\omega}_j \in F_2(\omega_i)\}$ be the set which contains all the states in the loop that share the same information set of F_2 as ω_i. Condition **i** implies that for every ω_i, the partition $A = D_i$ and $B = (D_i)^c$ satisfies $\#(A \to B) = \#(B \to A)$. Note that $|\{\omega_j; \omega_j \in F_2(\omega_i)\}| = \#(A \to B) + \#(A \to A)$, and $|\{\overline{\omega}_j; \overline{\omega}_j \in F_2(\omega_i)\}| = \#(B \to A) + \#(A \to A)$, where $\#(A \to A) = |\{i \in \{1, ..., m\}; \omega_i \in A,\ \overline{\omega}_i \in A\}|$. It follows from $\#(A \to B) = \#(B \to A)$ that

$$|\{\omega_j; \omega_j \in F_2(\omega_i)\}| = |\{\overline{\omega}_j; \overline{\omega}_j \in F_2(\omega_i)\}| \tag{6}$$

for every ω_i.

Define $J = \{i; \omega_i \in F_2(\overline{\omega}_i)\}$. We show that the rest of the states are decomposed into F_2-loops. Specifically, we show that if a finite set $S = \{(\omega_j, \overline{\omega}_j);\ \overline{\omega}_j \notin F_2(\omega_j)\}$, not necessarily an F_1-loop, satisfies Eq. (6) for every $\omega_i \in S$, then it is covered by F_2-loops.

When $|S| = 2$, Eq. (6) implies that this is an F_2-loop. We now assume the induction hypothesis: if Eq. (6) is satisfied for a set $S = \{(\omega_j, \overline{\omega}_j)\}$ and for every $\omega_i \in S$, and S contains less than or equal to m pairs, then it is covered by F_2-loops. We proceed by showing this statement for sets S containing $m + 1$ pairs.

We start at an arbitrary pair, say $(\omega_1, \overline{\omega}_1)$, and show that it belongs to an F_2-loop. Once this F_2-loop is formed, the states outside of this loop satisfy Eq. (6) for every ω_i outside of this loop. By the induction hypothesis, this set is covered by F_2-loops.

Due to Eq. (6), there is at least one $\overline{\omega}_j$ such that $\overline{\omega}_j \in F_2(\omega_1)$. Consider now the two pairs, $(\omega_j, \overline{\omega}_j, \omega_1, \overline{\omega}_1)$. If this is a loop, Eq. (6) remains true when applied to the states out of this loop. The induction hypothesis completes the argument. Otherwise, there is $\overline{\omega}_k$ where $k \neq 1, j$, such that $\overline{\omega}_k \in F_2(\omega_j)$. Consider now the three pairs, $(\omega_k, \overline{\omega}_k, \omega_j, \overline{\omega}_j, \omega_1, \overline{\omega}_1)$. If this is an F_2-loop, the other states satisfy Eq. (6), and as before, this set is covered by F_2-loops. However, if this is not an F_2-loop, Eq. (6) remains true, we annex another pair and continue this way until we obtain an F_2-loop. This loop might cover the entire set, but if not, the remaining states are, by the induction hypothesis, covered by F_2-loops. This shows **ii**.

ii $\Rightarrow$ **iii**. Let $f : \left\{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m\right\} \to (0, \infty)$ be a positive and F_2-measurable function. Suppose that $I_1, ..., I_r$ is a partition of $\{1, ..., m\}$, and for each $t = 1, ..., r$, the set $\Big((\omega_i, \overline{\omega}_i)\Big)_{i \in I_t}$ is an F_2-loop. Since, $\Big((\omega_i, \overline{\omega}_i)\Big)_{i \in I_t}$ is an F_2-loop, $\prod_{i \in I_t} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1$, which implies that

$$\prod_{i=1}^{m} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = \prod_{t=1}^{r} \prod_{i \in I_t} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1.$$

This proves **iii**. □

A.7 Proof of Proposition 2

Proof. Fix an F_i-loop $L_i = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I}$ where $I = \{1, 2, \ldots, m\}$. Let C_j denote the CKC that contains every pair $(\omega_j, \overline{\omega}_j)$.

Proof for first statement: Assume that L_i intersects the same CKC at least twice, so that $C_{l_1} = C_{l_2}$, where $l_1 < l_2$, is such CKC. Because L_i is a loop, the two pairs $(\omega_{l_1}, \overline{\omega}_{l_1})$ and $(\omega_{l_2}, \overline{\omega}_{l_2})$ that are in this CKC cannot be adjacent in the loop L_i, i.e., $l_1 \neq l_2 \pm 1$. Define the following sub-loop of L_i by omitting every state from $\overline{\omega}_{l_1}$ to ω_{l_2}. Formally, $L_i' = (\omega_1, \overline{\omega}_1, \ldots, \overline{\omega}_{l_1-1}, \omega_{l_1}, \overline{\omega}_{l_2}, \omega_{l_2+1}, \ldots, \omega_m, \overline{\omega}_m)$. This is a well-defined sub-loop of L_i (as $\omega_{l_1}, \overline{\omega}_{l_2} \in C_{l_1}$ while all other parts of the sub-loop match those of L_i), which implies that L_i is not irreducible. Note that the part we truncated from the loop L_i also forms a sub-loop, namely $L_i'' = (\omega_{l_2}, \overline{\omega}_{l_1}, \omega_{l_1+1}, \overline{\omega}_{l_1+1}, \ldots, \omega_{l_2-1}, \overline{\omega}_{l_2-1})$.

Proof for second statement: Assume, by contradiction, that L_i is irreducible, yet it has a pair of states $(\omega_l, \overline{\omega}_l)$ such that $\overline{\omega}_l \in F_i(\omega_l)$. This implies that $\{\overline{\omega}_{l-1}, \omega_l, \overline{\omega}_l, \omega_{l+1}\} \subseteq F_i(\omega_l) = F_i(\omega_{l+1})$. We can assume that $C_{l-1} \neq C_{l+1}$, otherwise the first statement suggests that L_i is not irreducible. So, define the following sub-loop of L_i by $L_i' = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I \setminus \{l\}}$. Note that L_i' is a well-defined sub-loop, as $C_{l-1} \neq C_{l+1}$ and $\overline{\omega}_{l-1} \in F_i(\omega_{l+1})$, thus contradicting the irreducible property.

Proof for third statement: Assume, w.l.o.g., that $F_i(\omega_1) \neq F_i(\overline{\omega}_1)$. If L_i intersects the same CKC twice, then we can follow the proof of the first statement, truncate the loop, and take a sub-loop that has an informative pair of states and intersects every CKC at most once. Thus, w.l.o.g., assume that L_i intersects every CKC at most once. Denote the set of informative pairs by $I^c = \{j : F_i(\omega_j) \neq F_i(\overline{\omega}_j)\}$ and define the following ordered sub-loop of L_i by $L_i' = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I^c}$. In simple terms, L_i' is generated from L_i by truncating all non-informative pairs $(\omega_j, \overline{\omega}_j)$, where $F_i(\omega_j) = F_i(\overline{\omega}_j)$, similarly to the process used in the proof of the second statement. Focusing on L_i', note that: (i) all pairs are pairwise disjoint; (ii) every CKC is crossed at most once; (iii) $\omega_{j+1} \in F_i(\overline{\omega}_j)$ as we removed only non-informative pairs; and (iv) $\omega_j \neq \overline{\omega}_j$ are both in the same CKC as in the original loop. Hence, L_i' is a well-defined loop and an F_i-fully-informative sub-loop of L_i.

Proof of fourth statement: If the loop L_i is irreducible, then the statement holds. Otherwise, it is not irreducible and we will prove by induction on the number of pairs m in

L_1. If $m = 2$, then L_i is irreducible. If $m = 3$ and L_i is not irreducible, then it has a sub-loop with two pairs. Assume w.l.o.g. that this sub-loop is based on the states $\{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2\}$. It cannot be that $F_i(\overline{\omega}_1) = F_i(\overline{\omega}_2)$, because that would make $(\omega_2, \overline{\omega}_2)$ a non-informative pair. So the sub-loop is $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ such that $F_i(\omega_1) = F_i(\overline{\omega}_2)$, but $F_i(\omega_1) = F_i(\overline{\omega}_3)$ and $F_i(\overline{\omega}_2) = F_i(\omega_3)$, so the pair $(\omega_3, \overline{\omega}_3)$ is non-informative.

Assume the statement holds for $m = k$ pairs, and consider an L_i loop with $k + 1$ pairs. If the loop intersects the same CKC more than once, we can split it into two sub-loops (as previously done), and use the induction hypothesis for each. Hence, we can assume that the loop does not intersect the same CKC twice.

Because the loop is not irreducible, there are two states ω_{i_1} and $\overline{\omega}_{i_2}$ that are not adjacent in the loop (so $i_1 \geq i_2 + 2$), yet $F_i(\omega_{i_1}) = F_i(\overline{\omega}_{i_2})$. The last equality also suggests that $F_i(\overline{\omega}_{i_1-1}) = F_i(\omega_{i_2+1})$. If $i_1 = i_2 + 2$, then there exists only one pair between the two states. This implies that the pair $(\omega_{i_2+1}, \overline{\omega}_{i_2+1}) = (\omega_{i_1-1}, \overline{\omega}_{i_1-1})$ is non-informative, contradicting the fact that L_i is F_i-fully-informative. So we conclude that $i_1 \geq i_2 + 3$. Define the following two loops $L_i' = (\omega_{i_1}, \overline{\omega}_{i_1}, \dots, \omega_{i_2}, \overline{\omega}_{i_2})$ and $L_i'' = (\omega_{i_2+1}, \overline{\omega}_{i_2+1}, \dots, \omega_{i_1-1}, \overline{\omega}_{i_1-1})$, where the ordering of states follows the original loop L_i. These are two well-defined F_i-loops with less than $k+1$ pairs each, so the induction hypothesis holds and the result follows.

If L_i does not intersect the same CKC more than once and does not have at least 4 states in the same partition element, then it is irreducible.

Proof of fifth statement: If the loop has a non-informative pair $\omega_j \in F_i(\overline{\omega}_i)$, then it contains 4 states from the same partition element, so assume that the loop is F_i-fully-informative and that it does not intersect the same CKC more than once. Thus, we need to prove that it has at least 4 states in the same partition element of F_i.

Consider the strict sub-loop L_i^- of L_i. It consists of pairs, taken from the original loop. Because L_i does not intersect the same CKC more than once, all the pairs of L_i^- are a strict subset of the pairs of L_i. This implies that some pairs were omitted from L_i when generating L_i^-, so assume w.l.o.g. that the pair $\{\omega_1, \overline{\omega}_1\}$ is not included in L_i^-. This implies that one pair $\{\omega_j, \overline{\omega}_j\}$ precedes in L_i^- a different one that it precedes in L_i. That is, $F_i(\overline{\omega}_j) = F_i(\omega_{j+1})$ according to L_i, whereas $F_i(\overline{\omega}_j) = F_i(\omega_k)$ where $k \neq j + 1$, according to L_i^-. But also $F_i(\omega_k) = F_i(\overline{\omega}_{k-1})$ according to L_i. Thus, $\{\overline{\omega}_j, \omega_{j+1}, \omega_k, \overline{\omega}_{k-1}\}$ are in the same partition element of L_i, as stated and the result follows. □

A.8 Proof of Theorem 5

Proof. The first statement follows similarly to the proof of Theorem 4, so we proceed to prove the second statement which is equivalent to the existence of a cover by loops of F_2.

Suppose, to the contrary, that an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is not F_2-balanced. This means that there is an F_2-measurable partition $\{A, B\}$ of these states such that Eq. (1) is not satisfied. Define an F_2-measurable signaling function that obtains two signals, α and β,

$$\tau_2(\alpha|\omega) = \begin{cases} x, & \text{if } \omega \in A, \\ y, & \text{if } \omega \in B, \end{cases} \tag{7}$$

and $\tau_2(\beta|\omega) = 1 - \tau_2(\alpha|\omega)$. On other states (outside the loop), τ_2 is defined arbitrarily. The numbers $x, y \in (0,1)$ are chosen so that $\frac{\ln x - \ln y}{\ln(1-x) - \ln(1-y)}$ is irrational.

Claim 1: If $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$, then any signal of τ_1 induces the same posteriors as α does or as β does in every CKC.

Claim 2: For any signal s of τ_1 and for any i, $\frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} \in \left\{\frac{x}{y}, \frac{1-x}{1-y}, \frac{y}{x}, \frac{1-y}{1-x}\right\}$. Therefore,

$$\prod_{i=1}^{m} \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = \left(\frac{x}{y}\right)^{\ell_1} \cdot \left(\frac{1-x}{1-y}\right)^{\ell_2} \cdot \left(\frac{y}{x}\right)^{k_1} \cdot \left(\frac{1-y}{1-x}\right)^{k_2},$$

where $\ell_1 + \ell_2 = |\{i; \omega_i \in A \text{ and } \overline{\omega}_i \in B\}|$ and $k_1 + k_2 = |\{i; \omega_i \in B \text{ and } \overline{\omega}_i \in A\}|$.

Claim 3: For any signal s of τ_1, $\prod_{i=1}^{m} \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = 1$.

We therefore obtain $(\frac{x}{y})^{\ell_1}(\frac{1-x}{1-y})^{\ell_2}(\frac{y}{x})^{k_1}(\frac{1-y}{1-x})^{k_2} = 1$. We conclude that there are whole numbers, say $\ell = \ell_1 - k_1$ and $k = k_2 - \ell_2$ such that $(\frac{x}{y})^{\ell} = (\frac{1-x}{1-y})^{k}$. Since $\frac{\ln x - \ln y}{\ln(1-x) - \ln(1-y)} = \frac{\ln \frac{x}{y}}{\ln \frac{1-x}{1-y}}$ is irrational, $\ell = k = 0$, implying that Eq. (1) is satisfied. This is a contradiction.

Moving on to the third part of the theorem, fix an irreducible F_1-loop L_1, and consider an cover with a unique F_2-loop L_2 which is irreducible, i.e., L_2 covers L_1 (up to the set of non-informative F_2 pairs) and both are irreducible.

Assume, by contradiction, that L_2 is not order-preserving. Denote $L_1 = (\omega_1, \overline{\omega}_1, \ldots, \omega_m, \overline{\omega}_m)$ and $L_2 = (\omega_1, \overline{\omega}_1, \omega_{i_2}, \overline{\omega}_{i_2}, \ldots, \omega_{i_m}, \overline{\omega}_{i_m})$. Thus, there exist indices $k > j > 1$ such that ω_k precedes ω_j in L_2. In simple terms, it implies that though L_2 covers a subset of pairs from L_1, the ordering of some pairs throughout the two loops differs, as suggested in Footnote 12.

Since the two loops are irreducible, it follows from Proposition 2 that they intersect every CKC at most once and that both are fully-informative. Moreover, for every state ω in every loop L_i, every set $F_i(\omega)$ contains two states from the loop L_i (otherwise, the loop is not irreducible). So, one can define an F_i-measurable function τ_i such that $\tau_i(s|\omega_l) = \tau_i(s|\overline{\omega}_{l-1}) \neq \tau_i(s|\omega_{l'})$ for every $\omega_l \neq \omega_{l'}$ in the loop.

To simplify the exposition, partition the states of L_2 into three disjoint sets: the set $A_1^2 = \{\overline{\omega}_1, \ldots, \omega_k\}$ contains all the states of L_2 from $\overline{\omega}_1$ till ω_k (following the order of L_2),

$A_k^2 = \{\overline{\omega}_k, \dots, \omega_j\}$ contains all the states of L_2 from $\overline{\omega}_k$ till ω_j, and $A_j^2 = \{\overline{\omega}_j, \dots, \omega_1\}$ which contains all remaining states of L_2. Follow a similar process with L_1, so that $A_1^1 = \{\overline{\omega}_1, \dots, \omega_j\}$ contains all the states of L_1 from $\overline{\omega}_1$ till ω_j (following the order of L_1), $A_j^1 = \{\overline{\omega}_j, \dots, \omega_k\}$ contains all the states of L_1 from $\overline{\omega}_j$ till ω_k, and $A_k^1 = \{\overline{\omega}_k, \dots, \omega_1\}$ which contains all remaining states of L_1.

Denote by C_l the CKC of the pair $(\omega_l, \overline{\omega}_l)$. Fix two distinct signals s_1 and s_2, and define the signaling function τ_2 as follows:

$$\tau_2(s_1|\omega) = 1 - \tau_2(s_2|\omega) = \begin{cases} p_1, & \text{if } F_2(\omega) \cap A_1^2 \neq \phi, \\ p_2, & \text{if } F_2(\omega) \cap A_k^2 \neq \phi, \\ p_3, & \text{if } F_2(\omega) \cap A_j^2 \neq \phi, \\ p_4, & \text{otherwise,} \end{cases}$$

where the probabilities $\{p_1, p_2, p_3, p_4\}$ are chosen as in the strategy defined in Lemma 1. Because the loop is irreducible, intersects every CKC at most once and F_2-fully-informative, τ_2 is a well-defined F_2-measurable function.

The result of Lemma 1 holds in every CKC of the loop (though with different probabilities). So given a CKC C_l, if there exists τ_1 such that $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$, then for every signal $t \in \text{Supp}(\tau_1)$ there exists a signal $s \in \{s_1, s_2\}$ and a constant $c > 0$ such that $\tau_1(t|\omega) = c\tau_2(s|\omega)$ for every $\omega \in C_l$. Therefore, in every CKC C_l and for every signal t, there exists a signal s such that $\frac{\tau_2(s|\omega_l)}{\tau_2(s|\overline{\omega}_l)} = \frac{\tau_1(t|\omega_l)}{\tau_1(t|\overline{\omega}_l)}$. Fix such a strategy τ_1.

Notice that in every CKC $C_l \neq C_1, C_j, C_k$ and for every signal $s \in \{s_1, s_2\}$, we get $\tau_2(s|\omega_l) = \tau_2(s|\overline{\omega}_l)$. Thus, $\frac{\tau_1(t|\omega_l)}{\tau_1(t|\overline{\omega}_l)} = 1$ for every t and every $l \neq i, j, k$. This implies that for every feasible signal t restricted to the loop L_1,

$$\tau_1(t|\omega) = \begin{cases} a_t, & \text{if } \omega \in A_1^1 = \{\overline{\omega}_1, \dots, \omega_j\}, \\ b_t, & \text{if } \omega \in A_j^1 = \{\overline{\omega}_j, \dots, \omega_k\}, \\ c_t, & \text{if } \omega \in A_k^1 = \{\overline{\omega}_k, \dots, \omega_1\}, \end{cases}$$

where $a_t, b_t, c_t \in (0, 1]$. Evidently, the parameters a_t, b_t, and c_t can vary across the feasible signals.

In addition, Lemma 1 states that in every CKC, $\tau_1(t|\omega)$ is proportional to $\tau_2(s_i|\omega)$ for some signal $s_i \in \{s_1, s_2\}$. This yields the following constraints:

$$\frac{\tau_1(t|\omega_1)}{\tau_1(t|\overline{\omega}_1)} = \frac{c_t}{a_t} = \frac{\tau_2(s_i|\omega_1)}{\tau_2(s_i|\overline{\omega}_1)} \in \left\{ \frac{p_3}{p_1}, \frac{1 - p_3}{1 - p_1} \right\},$$

$$\frac{\tau_1(t|\omega_j)}{\tau_1(t|\overline{\omega}_j)} = \frac{a_t}{b_t} = \frac{\tau_2(s_i|\omega_j)}{\tau_2(s_i|\overline{\omega}_j)} \in \left\{\frac{p_2}{p_3}, \frac{1-p_2}{1-p_3}\right\},$$

$$\frac{\tau_1(t|\omega_k)}{\tau_1(t|\overline{\omega}_k)} = \frac{b_t}{c_t} = \frac{\tau_2(s_i|\omega_k)}{\tau_2(s_i|\overline{\omega}_k)} \in \left\{\frac{p_1}{p_2}, \frac{1-p_1}{1-p_2}\right\}.$$

By Proposition 1, the fact that L_2 is F_1-covered implies that for any signal t in the support of τ_1, only two "types" of signals can appear: either all three ratios take the p-values, or all take the $(1-p)$-values (this follows from the uniqueness of the ratios, as stated in Lemma 1 in Appendix A.3). Let x (resp. y) be the total probability mass of the first (resp. second) signal-type at ω_1. Then τ_1's row-sum constraints at $\omega_1, \overline{\omega}_1$ and $\overline{\omega}_j$ yield

$$x + y = 1, \qquad \frac{p_1}{p_3}x + \frac{1-p_1}{1-p_3}y = 1, \qquad \frac{p_1}{p_2}x + \frac{1-p_1}{1-p_2}y = 1,$$

which has no solution because p_1, p_2, p_3 are distinct. Therefore L_2 must induce the same ordering of pairs as L_1, hence L_1 and L_2 coincide. □

A.9 Proof of Theorem 6

Proof. We first define an auxiliary set $\overline{\Omega}$, which groups together states that are in the same partition element of F_2 within CKCs. Formally, define the set $\overline{\Omega}$ such that $\eta(\omega') \in \overline{\Omega}$ if and only if $\eta(\omega') = \{\omega \in \Omega : \ \omega, \omega' \in C_j, \ F_2(\omega) = F_2(\omega')\}$. Accordingly, define the partition $\overline{F_2}$ to be discrete in every CKC, such that $\overline{F_2}(\eta(\omega)) = \overline{F_2}(\eta(\omega'))$ if and only if $F_2(\omega) = F_2(\omega')$. Note that $\overline{F_2}$ is essentially a projection of F_2 onto $\overline{\Omega}$. In addition, $\overline{F_1}$ is defined as follows: (i) discrete in every CKC, similarly to $\overline{F_2}$; (ii) $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ if ω and ω' are not in the same CKC, and there exist $\overline{\omega} \in \eta(\omega)$ and $\overline{\omega'} \in \eta(\omega')$ such that $F_1(\overline{\omega}) = F_1(\overline{\omega'})$; and (iii) $\overline{F_1}$ forms a partition (i.e., given (i) and (ii), if two elements of $\overline{F_1}$ contain the same state $\eta(\omega)$, they are unified into one element).

We now prove that $\overline{F_1} = \overline{F_2}$ in every CKC and that there are no $\overline{F_1}$-loops. Thus, by Theorem 4, any $\overline{F_2}$-measurable strategy $\overline{\tau_2}$ (which, extended to Ω, is also F_2-measurable) can be imitated by an $\overline{F_1}$-measurable strategy $\overline{\tau_1}$.

Step 1: $\overline{F_1} = \overline{F_2}$ in every CKC.

By definition, $\overline{F_2}$ refines $\overline{F_1}$, so we need to prove that $\overline{F_1}$ also refines $\overline{F_2}$ in every CKC. Assume, by contradiction, that $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ where ω and ω' are in the same CKC, whereas $\overline{F_2}(\eta(\omega)) \neq \overline{F_2}(\eta(\omega'))$. This suggests that $F_2(\omega) \neq F_2(\omega')$, which implies that $F_1(\omega) \neq F_1(\omega')$. According to the construction of $\overline{F_1}$, we conclude that the equality $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ followed from the partition-formation stage described in (iii) above, through at least one other CKC. Thus, there exists an F_1-loop which connects a state in $\eta(\omega)$ with a state in $\eta(\omega')$. Without loss of generality, assume these states are ω and ω'. Because every F_1-loop is F_2-non-informative, it follows that $F_2(\omega) = F_2(\omega')$, a contradiction.

Step 2: There are no $\overline{F_1}$-loops.

An $\overline{F_1}$-loop implies that an F_1-loop exists. By construction, all states within a single element of the partition $\overline{\Omega}$ in every CKC are F_2-equivalent (i.e., grouped together according to F_2). Because every F_1-loop is F_2-non-informative, it implies that the loop consists of only one $\overline{\Omega}$ state in every CKC, and not two. This contradicts the definition of a loop.

Step 3: $\overline{F_1}$ can mimic $\overline{F_2}$.

Fix a strategy τ_2, and let $\overline{\tau_2}$ be the projected strategy on $\overline{\Omega}$. Because $\overline{F_1} = \overline{F_2}$ in every CKC and there are no $\overline{F_1}$-loops, there exists an $\overline{F_1}$-measurable strategy $\overline{\tau_1}$ that imitates $\overline{\tau_2}$. Therefore, one can lift $\overline{\tau_1}$ to Ω to create τ_1, whose projection onto $\overline{\Omega}$ matches $\overline{\tau_1}$. Thus, the strategy τ_1 imitates τ_2, as needed. □

A.10 Proof of Proposition 3

Proof. Denote the two CKCs by C_1 and C_2. One part of the statement follows directly from Theorem 5, so assume that F_1 refines F_2 in every CKC and any F_1-loop is F_2-balanced. If there are no F_1-loops, then the result follows from Theorem 4, so assume there exists at least one F_1-loop, and every such loop is F_2-balanced.

Take any F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ with four states. We argue that either it is also an F_2-loop or it is F_2-non-informative. Otherwise, we can assume (without loss of generality) that $F_2(\omega_1) \neq F_2(\overline{\omega}_i)$, for every $i = 1, 2$. So, there are only two possibilities left: either $F_2(\omega_1) = F_2(\omega_2)$ or $F_2(\omega_1) \neq F_2(\omega_2)$. If $F_2(\omega_1) = F_2(\omega_2)$, then there exists an F_2-measurable partition of the four states such that $A = \{\omega_1, \omega_2\}$ and $B = \{\overline{\omega}_1, \overline{\omega}_2\}$, which is not balanced. Otherwise, there exists another non-balanced F_2-measurable partition of the form $A = \{\omega_1\}$ and $B = \{\overline{\omega}_1, \omega_2, \overline{\omega}_2\}$. In any case, we get a contradiction.

The proof now splits into two cases: either there exists an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ and an index i such that $F_2(\omega_i) \neq F_2(\overline{\omega_i})$, or every such loop is F_2-non-informative. If indeed every such loop is F_2-non-informative, Theorem 6 states that Oracle 1 dominates Oracle 2, so we need only focus on the former.

Assume that there exists an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ and an index i such that $F_2(\omega_i) \neq F_2(\overline{\omega}_i)$. Denote this couple by $\{\omega_1, \overline{\omega}_1\} \subseteq C_1$. The previous conclusion implies that it is also an F_2-loop. We claim that, under these conditions, every τ_2 is F_1-measurable. Note that F_1 refines F_2 in every CKC, so we need to verify that for every $(\omega, \overline{\omega}) \in C_1 \times C_2$ such that $F_1(\omega) = F_1(\overline{\omega})$, it follows that $F_2(\omega) = F_2(\overline{\omega})$.

Take $(\omega, \overline{\omega}) \in C_1 \times C_2$ such that $F_1(\omega) = F_1(\overline{\omega})$. If $\omega \in F_1(\omega_1)$ or $\omega = F_1(\overline{\omega}_1)$, then $(\omega, \overline{\omega})$ are part of the previously stated F_2-loop, so $F_2(\omega) = F_2(\overline{\omega})$. Otherwise, we can construct two new F_1-loops $(\omega, \overline{\omega}, \omega_1, \overline{\omega}_2)$ and $(\omega, \overline{\omega}, \omega_2, \overline{\omega}_1)$. Because $F_2(\omega_1) \neq F_2(\overline{\omega}_1)$, either $F_2(\omega) \neq F_2(\omega_1)$ or $F_2(\omega) \neq F_2(\overline{\omega}_1)$. The previous conclusion again implies that $(\omega, \overline{\omega})$ are part of an F_2-loop, so $F_2(\omega) = F_2(\overline{\omega})$, as needed. □

A.11 Proof of Theorem 7

Proof. We start by assuming that F_1 and F_2 are equivalent. According to Theorem 5, every F_i refines F_{-i} in every CKC, and every F_i-loop is covered by F_{-i}-loops. Fix an irreducible F_i-loop with at least 6 states, denoted L_i, and consider a cover by F_{-i}-loops. There are two possibilities: either the cover constitutes a single loop, or else. If the cover contains a shorter loop, say L'_{-i}, then that loop is not F_i-covered because L_i is irreducible, and this contradicts Theorem 5. Moreover, the cover cannot have non-informative pairs where $F_{-i}(\omega_i) = F_{-i}(\overline{\omega}_i)$, because the two partitions match one another in every CKC and L_i is irreducible. So, the cover consists of a single irreducible F_{-i}-loop, and Theorem 5 states that it is order-preserving. Thus, L_i and L_{-i} coincide as stated.

Moving to the other direction, let us prove that Oracle 1 dominates Oracle 2 (and the reverse dominance follows symmetrically). We start with two simple observations. First, in case F_1 has no loops, then the statement follows from previous results, so assume F_1 has loops. Second, we say that two CKCs C_1 and C_2 are connected if there exist $\omega_1 \in C_1$ and $\omega_2 \in C_2$ such that $F_1(\omega_1) = F_1(\omega_2)$. If there exists a CKC C which is not connected to any other CKC (i.e., for every $\omega \in C$, the partition element $F_1(\omega) \subseteq C$), then Oracle 1 dominates Oracle 2 conditional on that CKC and independently of all other CKCs. Thus, without loss of generality, we can assume that all CKCs are connected, either directly or sequentially.

For this part, we will need to define the notion of *type-2 irreducible* loops, which are fully-informative loops that do not have four states in the same information set of the relevant F_i.

Definition 9. *Let L_i be an F_i-loop. We say that the loop is* type-2 irreducible *if it does not have four states in the same information set (i.e., partition element) of F_i.*

We shall use this notion of type-2 irreducible F_1-loops as building blocks upon which every F_2-measurable τ_2 is also F_1-measurable. For that purpose, we start by proving in the following Claim 1 that every type-2 irreducible F_1-loop is also an F_2-loop. Next, we will extend this measurability result to every set of type-2 irreducible F_1-loops that intersect the same CKCs, and finally extend it to all CKCs that these loops intersect. These sets of CKCs, to be later defined as *clusters*, will be the basic sets on which every F_2-measurable strategy is also F_1-measurable.

Claim 1. *Every type-2 irreducible F_1-loop L_1 is an F_2-loop.*

Proof. If L_1 is irreducible, then it is also an irreducible F_2-loop, and the result holds. Thus assume that L_1 is not irreducible. Using the fifth result in Proposition 2, we deduce that L_1 intersects the same CKC more than once. Using the proof of the first result in Proposition 2, we can decompose L_1 into two disjoint strict sub-loops of F_1. This can be done repeatedly, so that L_1 is decomposed into sub-loops that do not intersect the same CKC more than once. This implies that every such loop is type-2 irreducible. Thus, every such sub-loop is irreducible, and so it is also an F_2-loop.

Note that the decomposition process occurs *within* every relevant CKC C and that $F_1|_C = F_2|_C$. That is, once there are two pairs of the same loop within the same CKC, we can decompose the loop into two disjoint loops by rearranging these four states. So, one can reverse the process and recompose the sub-loops of F_2 to regenerate the original loop L_1, which is now also an F_2-loop, as needed. □

Once we dealt with individual type-2 irreducible loops, we move to loops that intersect the same CKC. For that purpose, we need to prove the following Claim 2 which states that every F_i-fully-informative loop L_i can be decomposed to type-2 irreducible F_i-loops.

Claim 2. *Every F_i-fully-informative loop L_i that is not type-2 irreducible can be decomposed to type-2 irreducible F_i-loops.*

Proof. Let $L_i = (\omega_1, \overline{\omega}_1, \ldots, \omega_m, \overline{\omega}_m)$ be F_i-fully-informative and not type-2 irreducible. Then some atom of F_i contains four distinct loop states $\{\overline{\omega}_j, \omega_{j+1}, \overline{\omega}_\ell, \omega_{\ell+1}\}$ with $\ell > j+1$ (the indices cannot be adjacent because L_i is fully-informative). Cut L_i at these two occurrences and reconnect inside that atom to obtain two shorter F_i-fully-informative loops,

$$L^1 = (\omega_j, \overline{\omega}_j, \omega_{\ell+1}, \overline{\omega}_{\ell+1}, \ldots, \omega_{j-1}, \overline{\omega}_{j-1}), \qquad L^2 = (\omega_\ell, \overline{\omega}_\ell, \omega_{j+1}, \overline{\omega}_{j+1}, \ldots, \omega_{\ell-1}, \overline{\omega}_{\ell-1}).$$

Each has strictly fewer pairs than L_i. Iterating this splitting procedure terminates and yields a decomposition into type-2 irreducible F_i-loops. Moreover, if an F_i-atom contains at most two states of L_i, its corresponding F_i-connection can be preserved inside one of the resulting loops throughout the decomposition. □

Using Claim 2, we now prove in the following Claim 3, that every F_2-measurable strategy on two type-2 irreducible F_1-loops with a joint CKC (i.e., pass through the same CKC) is F_1-measurable.

Claim 3. *Fix two type-2 irreducible F_1-loops L_1 and L_1' that share at least one* CKC. *Then, every $\tau_2|_{L_1 \cup L_1'}$ is F_1-measurable.*

Proof. Assume, toward a contradiction, that $\tau_2|_{L_1 \cup L_1'}$ is F_2-measurable but not F_1-measurable. Since L_1 and L_1' are also F_2-loops, there exist states $\omega \in L_1$ and $\omega' \in L_1'$ with $F_1(\omega) = F_1(\omega')$ but $F_2(\omega) \neq F_2(\omega')$. Because F_1 and F_2 coincide within every CKC, ω and ω' must lie in different CKCs.

Relabel the loops cyclically so that $\omega = \omega_1$ in $L_1 = (\omega_1, \overline{\omega}_1, \ldots, \omega_m, \overline{\omega}_m)$ and $\omega' = \overline{\omega}_1'$ in $L_1' = (\omega_1', \overline{\omega}_1', \ldots, \omega_{m'}', \overline{\omega}_{m'}')$. Let C be a CKC visited by both loops, and pick an index j such that $\{\omega_j, \overline{\omega}_j\} \subseteq C$ and an index j' such that $\{\omega_{j'}', \overline{\omega}_{j'}'\} \subseteq C$. Consider the sequence of pairs obtained by following L_1 up to ω_j, then *switching* inside C by pairing ω_j with $\overline{\omega}_{j'}'$, and finally following L_1' until the pair $(\omega_1', \overline{\omega}_1')$: $(\omega_1, \overline{\omega}_1, \ldots, \omega_j, \overline{\omega}_{j'}', \omega_{j'+1}', \overline{\omega}_{j'+1}', \ldots, \omega_1', \overline{\omega}_1')$. This is an F_1-loop (after deleting any consecutive non-informative pair, if it arises) and it closes because $F_1(\omega_1) = F_1(\overline{\omega}_1')$. If it is type-2 irreducible, then by Claim 1 it is also an F_2-loop, implying $F_2(\omega_1) = F_2(\overline{\omega}_1')$, a contradiction.

Otherwise, apply Claim 2 to decompose it into type-2 irreducible F_1-loops. By the last sentence of that claim, we can carry out the decomposition so that the closing F_1-connection between ω_1 and $\overline{\omega}_1'$ is preserved in one of the resulting loops, which then yields the same contradiction. [16] □

Next, we extend the result of Claim 3 to more than two loops. Specifically, we say that two loops L_i and L_i' are *connected* if either they share at least one CKC, or there exists a sequence of loops starting with L_i and ending with L_i' where each two consecutive loops share at least one CKC.

Claim 4. *Consider a set A of type-2 irreducible and connected F_1-loops, i.e., every two loops are connected by some of these type-2 irreducible loops. Then, every F_2-measurable $\tau_2|_A$ is F_1-measurable.*

Proof. Let us prove this by induction on the number of loops. The case of two loops is proved in Claim 3, so assume the statement holds for m loops, and consider a set of $m+1$ type-2 irreducible and connected F_1-loops. Further assume, by contradiction, that there exists an F_2-measurable strategy over this set that is not F_1-measurable. Thus, there exists ω and ω' such that $F_2(\omega) \neq F_2(\omega')$ whereas $F_1(\omega) = F_1(\omega')$. Evidently, ω and ω' are in different loops and different CKCs. Denote the loops of ω and ω' by L_1 and L_1', respectively.

If L_1 and L_1' are connected directly (through a joint CKC) or through at most m loops (including L_1 and L_1'), then the induction hypothesis holds and every F_2-measurable strategy on this set of loops is F_1-measurable, implying that $F_2(\omega) = F_2(\omega')$. Thus, assume that L_1 and L_1' are connected through a sequence of all the $m+1$ loops (including L_1 and L_{m+1}).

[16] The same construction works for any ω, ω' lying in two different CKCs visited by connected loops: if $F_1(\omega) = F_1(\omega')$, one can build an F_1-fully-informative loop linking them and conclude $F_2(\omega) = F_2(\omega')$.

Note that ω' cannot be in the same partition element as any other state from this set of loops, other than ω, the state connected to ω in L_1, and the state connected to ω' in L'_1. Otherwise, either one of these loops is not type-2 irreducible, or the F_2-measurability constraints with every intermediate loop is met (by the induction hypothesis), and again we get that $F_2(\omega) = F_2(\omega')$.

Thus, we can now follow the same stages as in the proof of Claim 3 and generate an F_1-fully-informative loop based on the sequence of loops connecting L_1 and L'_1 (as well as ω and ω'), which starts with ω_1 and ends with $\overline{\omega}'_1$. In this case, Claim 2 holds and we get a type-2 irreducible F_1-loop, which starts with ω_1 and ends with $\overline{\omega}'_1$, that is also an F_2-loop. We therefore conclude that $F_2(\omega) = F_2(\omega')$ and the induction follows accordingly. □

After we established that every F_2-measurable strategy over a set of connected loops is F_1-measurable, let us extend this result to all the CKCs that these loops intersect. For that purpose, let A be a maximal set of connected loops, where every two are connected, and let C_A be the set of all CKCs that intersect one of these loops (that is, every CKC contains a pair of states from one of these loops). We refer to every C_A as a *cluster*. We argue that every F_2-measurable strategy over a cluster C_A is F_1-measurable. To see this, recall Footnote 16 which states that the proof of Claim 3 holds for every ω and ω' in two different CKCs that intersect two connected loops L_1 and L'_1, respectively. Namely, for every two such states ω and ω' where $F_1(\omega) = F_1(\omega')$, it follows that $F_2(\omega) = F_2(\omega')$. So, as argued in the proof of Claim 4, we conclude that every F_2-measurable strategy over a cluster is F_1-measurable.

Observation 2. *Every F_2-measurable strategy over a cluster is F_1-measurable.*

Once we have established that every F_2-measurable strategy over a cluster is F_1-measurable, let us consider a partition Ω^* of Ω into clusters and individual CKCs that are not part of clusters. Note that *any* two elements of the partition Ω^* jointly intersect at most one partition element of F_1, otherwise the two components would be in the same cluster. To see this, consider the different possible intersections of elements in Ω^*. If both elements A_1 and A_2 are CKCs, then any two different partition elements of F_1 that intersect both A_1 and A_2 would form a type-2 irreducible F_1-loop. Otherwise, one of these elements is a cluster, say A_1, and it follows from previous proofs that for every ω and ω' that belong to the same cluster (but in different CKCs) and $F_1(\omega) = F_1(\omega')$, then one can form an F_1-fully-informative loop that starts with ω and ends with ω'. Thus, in case ω and ω' are in cluster A_1 and in different partition elements of F_1 that intersect A_2 (whether A_2 is a CKC or another cluster), one can form an F_1-fully-informative loop that intersects A_1 and A_2. Using Claim 2, we can conclude that A_1 and A_2 belong to the same cluster. This result is summarized in the following observation.

Observation 3. *Fix two elements $A_1, A_2 \in \Omega^*$. Then, there exists at most one partition element $F_1(\omega)$ of F_1 such that $F_1(\omega) \cap A_1$ and $F_1(\omega) \cap A_2$ are non-empty sets.*

We would now want to prove that Oracle 1 can mimic every F_2-measurable strategy defined over Ω^*. For this purpose, we present the following Lemma 2 which relates to the F_2-measurability constraints over different sets of CKCs, that are not in the same cluster (i.e., they are not connected by type-2 irreducible F_1-loops).

Lemma 2. *Fix two disjoint sets $A_1, A_2 \subseteq \Omega^*$ that do not intersect the same* CKCs*, and denote $A = A_1 \cup A_2$. Assume that:*

- *For every i and for every F_2-measurable $\tau_2|_{A_i}$, there exists an F_1-measurable $\tau_1^i|_{A_i}$, such that $\mu_{\tau_1^i}|_{A_i} = \mu_{\tau_2}|_{A_i}$.*
- *For every $\omega_1, \omega_1' \in A_1$ and $\omega_2, \omega_2' \in A_2$ such that $F_1(\omega_1) = F_1(\omega_2)$ and $F_1(\omega_1') = F_1(\omega_2')$, it follows that $F_1(\omega_1) = F_1(\omega_1')$.*

Then, for every $\tau_2|_A$, there exists $\tau_1|_A$ such that $\mu_{\tau_1}|_{A_i} = \mu_{\tau_2}|_{A_i}$ for every $i = 1, 2$.

Proof. Fix $\tau_2|_A$ and $\tau_1^i|_{A_i}$ where $i = 1, 2$, such that $\mu_{\tau_2}|_{A_i} = \mu_{\tau_1^i}|_{A_i}$ for every i. Define the sets $\tilde{A}_i = \{\omega_i \in A_i : \exists \omega_{-i} \in A_{-i}, F_1(\omega_i) = F_1(\omega_{-i})\}$ for every $i = 1, 2$. The second condition of the claim implies that all the states in $\tilde{A}_1 \cup \tilde{A}_2$ are in the same partition element of F_1. To see this, fix $\omega_1 \in \tilde{A}_1$ and, by definition, there exists a state $\omega_2 \in \tilde{A}_2$ such that $F_1(\omega_1) = F_2(\omega_2)$. If there exists another $\omega_1' \in \tilde{A}_1$, it is either connected to ω_2 (i.e., $F_1(\omega_1') = F_1(\omega_2)$), or to some $\omega_2' \in \tilde{A}_2$, and in that case the condition implies that $F_1(\omega_1) = F_1(\omega_1')$. The same holds for every $\omega_2 \in \tilde{A}_2$.

For every $i = 1, 2$, let S_i be the signals induced by $\tau_1^i|_{A_i}$. Define the following strategy τ_1:

$$\tau_1((s_1, s_2)|\omega) = \begin{cases} \tau_1^1(s_1|\omega)\tau_1^2(s_2|\tilde{A}_2), & \text{if } \ \omega \in A_1, (s_1, s_2) \in S_1 \times S_2, \\ \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\omega), & \text{if } \ \omega \in A_2, (s_1, s_2) \in S_1 \times S_2. \end{cases}$$

One can easily verify that $\sum_{(s_1,s_2)} \tau_1((s_1, s_2)|\omega) = 1$ for every ω, so τ_1 is indeed a strategy.

Let us now prove that τ_1 is F_1-measurable and $\mu_{\tau_1}|_A = \mu_{\tau_2}|_A$. If we restrict τ_1 to A_i, it is clearly F_1-measurable as $\tau_1^{-i}(s_{-i}|\tilde{A}_{-i})$ is fixed for every $\omega \in A_i$ and $s_i \in S_i$. Thus, consider $\tau_1((s_1, s_2)|\omega)$ where $\omega \in \tilde{A}_1$. All the states in $\tilde{A}_1 \cup \tilde{A}_2$ are in the same partition element of F_1, so for every $(\omega_1, \omega_2) \in \tilde{A}_1 \times \tilde{A}_2$ we get

$$\begin{aligned} \tau_1((s_1, s_2)|\omega_1) &= \tau_1^1(s_1|\omega_1)\tau_1^2(s_2|\tilde{A}_2) = \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\tilde{A}_2) \\ &= \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\omega_2) = \tau_1((s_1, s_2)|\omega_2), \end{aligned}$$

and the F_1-measurability condition holds. Moreover, for every $\omega_i, \omega_i' \in A_i$ and for every (s_1, s_2) such that $\tau_1^i(s_i|\omega) > 0$ where $\omega \in \{\omega_i, \omega_i'\}$, it follows that

$$\frac{\tau_1((s_1,s_2)|\omega_i, A_i)}{\tau_1((s_1,s_2)|\omega_i', A_i)} = \frac{\tau_1^i(s_i|\omega_i)}{\tau_1^i(s_i|\omega_i')},$$

which implies that conditional on A_i, τ_1 yields the same distribution over posteriors profiles as τ_1^i, thus mimicking τ_2 on every A_i, as needed. □

We can thus finalize the proof using induction on the number of elements in Ω^*. Until now, we established in Observation 2, Observation 3 and Lemma 2 that, given either $|\Omega^*| = 1$ or $|\Omega^*| = 2$, then for every F_2-measurable strategy $\tau_2|_{\Omega^*}$, there exists $\tau_1|_{\Omega^*}$ such that $\mu_{\tau_1}|_A = \mu_{\tau_2}|_A$ for every $A \in \Omega^*$. Assume this holds for $|\Omega^*| = k \geq 2$, and consider $|\Omega^*| = k + 1$.

Denote the elements of Ω^* by $A_1, A_2, \ldots, A_k, A_{k+1}$. If there exists only one partition element of F_1 that intersects A_{k+1} and at least one A_i for $i \leq k$, then Lemma 2 holds and the result follows. Thus, assume there are at least two different partition elements $F_1(\omega) = F_1(\omega_1)$ and $F_1(\omega') = F_1(\omega_2)$ of F_1 such that $\omega, \omega' \in A_{k+1}$ and $\omega_i \in A_i$ for every $i = 1, 2$.

The proof now splits into two parts: either A_1 and A_2 are connected (i.e., there exists a sequence of partition elements of F_1 that sequentially intersect elements in $\Omega^* \setminus A_{k+1}$, starting with A_1 and ending with A_2) or A_1 and A_2 are unconnected. If they are unconnected, we can apply Lemma 2 for A_1 and A_{k+1} and then use the induction hypothesis, so we assume they are connected.

Whether A_{k+1} is a CKC or a cluster and assuming that A_1 and A_2 are connected, we argue that there exists a type-2 irreducible F_1-loop that include ω and ω', implying that A_{k+1} is part of a cluster with other elements in Ω^*. To see this, recall whenever ω and ω' belong to the same cluster and $F_1(\omega) = F_1(\omega')$, then there exists an F_1-fully-informative loop that start with ω and ends with ω'. So consider such a sequence of states $l_{\omega \to \omega'} = (\omega, \ldots, \omega')$, which would have been an F_1-loop had $F_1(\omega) = F_1(\omega')$.

Next, fix the entire path of connections of elements in Ω^* that starts with A_1 and ends with A_2. Again, the connection between A_1 and A_2 implies that there exists a sequence of states $l_{\omega_1 \to \omega_2} = (\omega_1, \ldots, \omega_2)$ in $\Omega^* \setminus A_{k+1}$, that would have been an F_1-loop had $F_1(\omega_1) = F_1(\omega_2)$. Hence, consider the sequence of states $l = (\omega, \ldots, \omega', \omega_2 \ldots, \omega_1)$ which forms an informative F_1-loop, because $F_1(\omega) \neq F_1(\omega')$. Using Proposition 2 and Claim 2, we know that this loop has a type-2 irreducible F_1-sub-loop that contains ω and ω'. Thus, A_{k+1} is in the same cluster as other elements in Ω^*, thus contradicting the assumption that $|\Omega^*| = k + 1$. □