Mathematics for Financial Economics

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This booklet was written as class notes for the course "Mathematics for M.A. in Financial Economics", given by David Lagziel at The Interdisciplinary Center as of October 2015. The course is intended to provide the students with the relevant mathematical tools for their M.A studies in financial economics. The material covers topics in Calculus, Linear Algebra, Probability, and Statistics. All students are assumed to have background in undergraduate level of these courses. Thus, we do not presume to cover all possible studied material, but review it briefly with an emphasis on mathematical techniques. The notes may contain errors and misprints, and therefore should be taken with limited liability. Any comments, suggestions, and\or corrections are welcomed at Dudulagziel@gmail.com. **Last update: August 23, 2018**.

Contents

CHAPTER 1

Introduction

Math is the basic language that we use in economics. In order to understand the way people, prices, firms and more operate we need to use basic and advanced math. The purpose of this course and book is to give M.A. students in economics the basic tools to deal with economically relevant problems. The knowledge of mathematics is not a privilege for economists but a duty. An economist without good understanding of the mathematical tools will not be able to correctly model the problem he wishes to approach at the first place, without even going into the process of solving it. Thus, every good economist should have a good understanding of mathematics.

This book is divided into 4 parts. The first part discusses one-variable calculus, which you probably learned in your first year of your undergraduate degree. The second part is Linear Algebra, that (again) most of you already know. Although both parts are relatively simple, we try and give as many economically relevant examples as we can, so they will still be interesting. The third parts concerns mulch-variables calculus. This part is a bit more advanced and requires a good knowledge of basic calculus. The last part discusses probability and statistics, and although it might not seem related to the other chapters, one can still find a few good connections.

Clearly there are more subjects that we will not go over in this book, such as differential equations. This does not mean these fields are not important. For additional information, one can use the two main book used for this course, which are:

(1) "Mathematics for Economists" by Carl P. Simon and Lawrence Blume.
(2) "Must Have Tools for Graduate Study in Economics" by William Neilson.

Part 1

Basic Calculus

CHAPTER 2

One-variable calculus

2.1. Functions in $\mathbb{R}^1$

2.1.1. Sets and intervals.

We start with a few of the basic elements in math - numbers and functions. The numbers are usually categorized into the following sets:

- $\mathbb{R}$ is the set of all real numbers and it contains all numbers that are not complex. That is $\{1, 2, \pi, e, \sqrt{2}\} \subsetneq \mathbb{R}$.
- $\mathbb{Z}$ is the set of all integers, i.e., $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbb{N}$ is the set of all natural numbers (which are non-negative integers) $\{0, 1, 2, 3, \dots\}$.
- $\mathbb{Q}$ is the set off all rational numbers, that is numbers that can be represented as a fraction of two integers when the denominator does not equal zero. For example, $1 \in \mathbb{Q}$, $\frac{1}{3} \in \mathbb{Q}$ and so on.

Most students usually have difficulty with distinguishing between real numbers and rational numbers. First, note that the set of rational numbers is a subset of the real numbers. Moreover, all the previously defined sets are subsets of $\mathbb{R}$. However, the latter is not a subset of the other sets. For example, $\sqrt{2}$ is a real number but it is not a rational number and it is not an integer. We will prove this later on. In addition, the relations between the sets are

$$\mathbb{N} \subsetneq \mathbb{Z} \subsetneq \mathbb{Q} \subsetneq \mathbb{R}.$$

There are other types of numbers and sets, but we will not require them in this course.
One specific type of commonly used sets are sets with with a continuum of numbers, referred to as *intervals*. Let $a, b \in \mathbb{R}$ be two real numbers such that $a < b$. The set of all real numbers between a and b is called an *interval*. Any interval can be finite or infinite, and can contain the end points or not, for example:

$$\begin{aligned}
[a,b] &= \{x \in \mathbb{R} : \; a \le x \le b\}, \\
(a,b) &= \{x \in \mathbb{R} : \; a < x < b\}, \\
(a,b] &= \{x \in \mathbb{R} : \; a < x \le b\}, \\
[a,b) &= \{x \in \mathbb{R} : \; a \le x < b\}, \\
(-\infty,a] &= \{x \in \mathbb{R} : \; x \le a\}, \\
(b,\infty) &= \{x \in \mathbb{R} : \; b < x\},
\end{aligned}$$

and there are many other examples.

2.1.2. Basic and advanced functions. After establishing the required sets of numbers, we move on to functions. Functions are mathematical objects that transform elements from one set to elements in another set. This is a rather vague description, therefore we limit our scope only to *real-valued functions*. A real-valued function $f : \mathbb{R} \to \mathbb{R}$ basically takes a number $x \in \mathbb{R}$ from the *domain* and transforms it into a number $f(x) \in \mathbb{R}$ in the *co-domain*. All the values $f(x)$ define the image of f, i.e., the image of f is given by $\{f(x) \in \mathbb{R} : \; x \in \mathbb{R}\}$.[1] The input variable x is an independent variable, and in economic applications is also called an *exogenous variable*, and the output variable $f(x)$ is called the dependent

[1]The term *range* refers to either the co-domain, or the image. To avoid confusion, we will generally avoid this term.

variable (an *endogenous variable*). As you are probably well aware of the concept of functions, we can go over a few classes of commonly-used functions.

(1) **Polynomials.** A polynomial $p(x)$ is a function of the form

$$p(x) = a_0 x^0 + a_1 x^1 + \cdots + a_n x^n$$

when $\{a_i : i = 0, 1, \ldots, n\} \subset \mathbb{R}$ and all indices 0 through n are natural numbers. When $a_n \neq 0$, the function $p(x)$ we described is also called a polynomial function of degree n. Note that the degree relates to highest index i whose weight a_i is not equal to 0. The polynomials are a wildly used class of functions. For example, every constant function $f(x) = c$ (when $c \in \mathbb{R}$ is a constant) is a polynomial. Every linear function $f(x) = ax + b$ when $a \neq 0$ is a polynomial of degree one. A function of the form $ax^2 + bx + c$ when $a \neq 0$ is called a parabolic function and so on.

(2) **Rational functions.** A rational function $R(x) = \frac{p(x)}{q(x)}$ is a function given by a ratio of two polynomials. For example,

$$R(x) = \frac{1 - x^3}{1 + x^2}, \; R(x) = \frac{x + x^2}{7 - x^2}, \; R(x) = \frac{9x^5 - 2}{x}.$$

(3) **Trigonometric functions.** The basic trigonometric functions are $\sin(x)$, $\cos(x)$ and $\tan(x)$. The trigonometric functions are defined through ratios between different edges of a right triangle (*a rectangle triangle)* as described in Figure 2.1.1

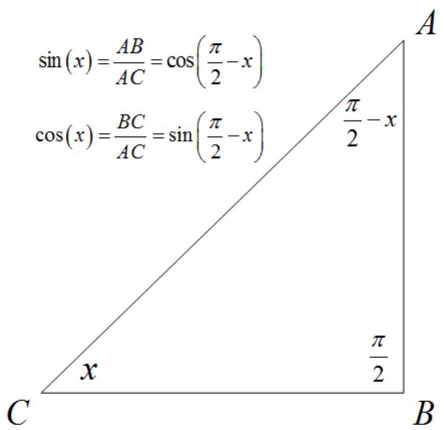

FIGURE 2.1.1. Definition of the $\sin(x)$function and $\cos(x)$function.

The definition of $\tan(x)$ is $\tan(x) = \frac{\sin(x)}{\cos(x)}$. These functions are usually defined on the unit circle and thus have many special properties such as periodicity, for example.

(4) **Exponential functions.** For every $a > 0$, let $f(x) = a^x$ be an exponential function with base a. Usually we assume that $a \neq 1$, since the function becomes a constant function, meaning a polynomial of degree 0.

(5) **Inverse function.** In many cases, given a function $f(x)$, one can define the inverse function of f, denoted by $f^{-1}(x)$. The inverse function does the opposite to what f does by taking $f(x)$and transforming it to x. The inverse function is not always well-defined, as the original function needs to be a one-to-one mapping. That is, every value $f(x)$ that the function can reach, must have a unique x that generates it. For example, consider the function $f : D \to \mathbb{R}$ where $D = \{1, 2, 3\}$ such that

$$\begin{aligned} f(1) &= 6, \\ f(2) &= \pi, \\ f(3) &= 6. \end{aligned}$$

Such function does not have an inverse as it is not a *one-to-one* mapping from D to $\mathbb{R}$. Namely, the inverse could states that π translates to 2, but we have a problem with the value 6, as f sends both 1 and 3 to 6.

(6) **Logarithmic functions.** A good example of an inverse function is the $\log_a(x)$. This function is the inverse function to a^x. It is defined through the exponential function such that, if $a^x = y$, then $\log_a(y) = x$. The logarithmic function has many properties that you should know, such as:
   (a) $\log_a(xy) = \log_a(x) + \log_a(y)$.
   (b) $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$.
   (c) $\log_a(x^y) = y\log_a(x)$.
   (d) $a^{\log_a(x)} = x$.
   (e) $\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$.

REMARK 2.1. There is one important distinction that should be made clear. The notation we use $(x, f(x))$ are general and are sometimes confused with the (x, y) notation as the value of the function is denoted by y. The origin of this notation is in the way graphs of functions are drawn. One variable functions are drawn on the (x, y) plain, $\mathbb{R}^2$, and most of the times the axes are denoted by x and y. Thus, sometimes there is a confusion between the notation of the value of the function in a specific point and the notation of the axes.

2.1.3. Useful economic functions. There are several functions that are commonly used in economics.

- The *demand function* $D(x)$ sets the price $p = D(x)$ charged for each unit when x units are sold.
- The *supply function* $S(x)$ sets the price $p = S(x)$ for which producers will supply x units.
- The *cost function* $C(x)$ determines the cost of producing x units.
- The *revenue function* $R(x)$ defines the revenue from selling x units, and given by $R(x) = xD(x)$.
- The *profit function* $P(x)$ defines the net profit for selling x units. Given by

$$P(x) = R(x) - C(x) = xD(x) - C(x).$$

EXERCISE 2.1. The total cost of producing x units is given by $C(x) = x^3 - 2x^2 + \frac{6}{x}$.

(1) Find the domain of C.
(2) What is the marginal cost for producing the 3rd unit?

Solution.

(1) The domain is $\{1, 2, 3, \dots\}$.
(2) The cost for producing 2 units is $C(2) = 8 - 8 + 3 = 3$. The cost for producing 3 units is $C(3) = 27 - 18 + 2 = 11$. Thus, the 3rd unit's marginal cost is $C(3) - C(2) = 11 - 3 = 8$.

EXERCISE 2.2. The total cost of producing x units is given by $C(x) = 2^x - x^2$. On every workday $x(t) = 3t - 1$ units are manufactured in the first t hours.

(1) How much will be spent on production by the end of the third hour?
(2) What is the minimal number of hours such that the manufacturing cost exceeds \$1000?

Solution.

(1) We need to observe the composition of C and x as a function of t. Specifically, we get

$$\begin{aligned} C(x(t)) &= 2^{x(t)} - (x(t))^2 \\ &= 2^{3t-1} - (3t-1)^2. \end{aligned}$$

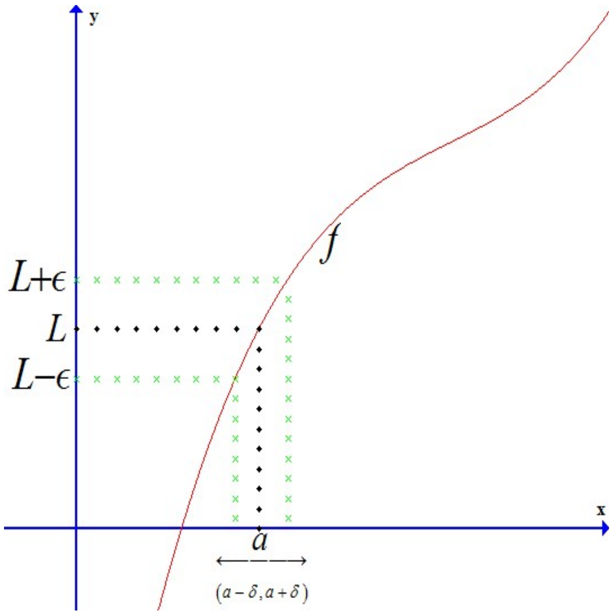

FIGURE 2.2.1. The limit of a function f when $x \to x_0$.

Thus,

$$C\left(x\left(3\right)\right) = 2^{9-1} - \left(9-1\right)^2 = 256 - 64 = 192.$$

(2) We need to find the minimal t such that $2^{3t-1} - \left(3t-1\right)^2 > 1000$. Since this inequality is difficult to solve accurately, we can simply check the production cost in the following hours.

$$C\left(x\left(4\right)\right) = 2^{12-1} - \left(12-1\right)^2 = 2048 - 121 = 1927,$$

and the answer is $t = 4$ hours.

2.2. Limits & continuous functions

2.2.1. Limits of one-variable functions.

The first property we consider is whether a function is continuous or not. Though this property is quite intuitive and clear, the definition is a bit more complex. We start with the definition of a finite limit at a finite point.

DEFINITION 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a one-variable function and let $x_0, L \in \mathbb{R}$ be real numbers (we sometimes denote the domain of f by D and the co-domain by $f\left(D\right)$). L is the limit of f in the point x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $|x - x_0| < \delta$, it follows that

$$|f\left(x\right) - L| < \epsilon.$$

We denote this limit by $\lim_{x \to x_0} f\left(x\right) = L$.
In simple words, L is the limit of the function f when x tends to x_0 if f can get sufficiently close to L (with an infinitely small deviation of no more than ϵ, for any $\epsilon > 0$) when x is close to x_0. See Figure 2.2.1.

REMARK 2.2. one can write the term $|x - x_0| < \delta$, in the following manner $x \in \left(x_0 - \delta, x_0 + \delta\right)$, which tends to be more convenient to students.

Definition 2.1 only relates to finite values of x_0 and L. There are many cases that are not included in this definition, specifically, when L and\or x_0 are infinite. We will give three more definitions for some of the additional cases.

DEFINITION 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be a one-variable function and let $x_0, L \in \mathbb{R} \cup \{-\infty, +\infty\}$ be real numbers (including infinity and minus infinity).

- **Case I** - $x_0 = \infty$ and $L = \infty$. We say that L is the limit of f when $x \to x_0$ if for every $M > 0$ there exists a $K > 0$ such that for every $x > K$, it follows that $f(x) > M$.
- **Case II** - x_0 is finite and $L = \infty$. We say that L is the limit of f when $x \to x_0$ if for every $M > 0$ there exists a $\delta > 0$ such that for every $|x - x_0| < \delta$, it follows that $f(x) > M$.
- **Case III** - $x_0 = \infty$ and L is finite. We say that L is the limit of f when $x \to x_0$ if for every $\epsilon > 0$ there exists a $K > 0$ such that for every $x > K$, it follows that $|f(x) - L| < \epsilon$.

The idea behind an infinite limit is as follows. The limit of a function is infinite if its values are becoming larger and larger and unbounded as you approach a finite x_0 or as $x \to \infty$. These cases do not include all possible cases as we only considered positive values and did not relate to negative ones. You could find additional definitions in most academic math books. For example, the limit of $f(x) = x^2 - 3$ is ∞ when $x \to \infty$ and when $x \to -\infty$. In addition, although the function $f(x) = 1/x$ is not defined when $x = 0$, its limits when $x \to 0$ is ∞. This is another important aspect. The limit is independent of the value of the function in x_0, hence the function is not necessarily defined in x_0, yet the limit exists.

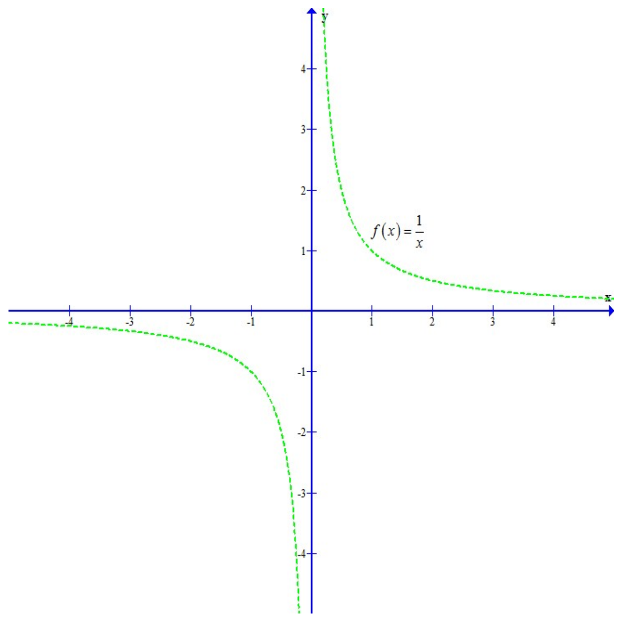

FIGURE 2.2.2. The function $f(x) = \frac{1}{x}$ and its one-sided limits when $x \to 0^{\pm}$.

REMARK 2.3. A limit does not always exist. There are cases when the definition fails and then we say that the limit does not exist. A good, but a bit more complicated example of a limit that does not exist the limit of the function $\sin\left(\frac{1}{x}\right)$ when $x \to 0$. See Figure 2.2.2 and Figure 2.2.3.

2.2.2. Limits properties.

The arithmetic limit laws are the following: Assume that both limits converge, $\lim_{x\to x_0} f(x) = a$, $\lim_{x\to x_0} g(x) = b$ (including the case where x_0 is possibly $\pm\infty$), then

- $\lim_{x\to x_0} (f(x) \pm g(x)) = a \pm b$;
- $\lim_{x\to x_0} (f(x) \cdot g(x)) = a \cdot b$;
- $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{a}{b}$, assuming that $g(x), b \neq 0$.

EXERCISE 2.3. The number of produced units per month as a function of t (in months) is $P(t) = \frac{6t^2+5t}{(t+1)^2}$. Determine what we will the the long run production level (i.e., when $t \to \infty$).

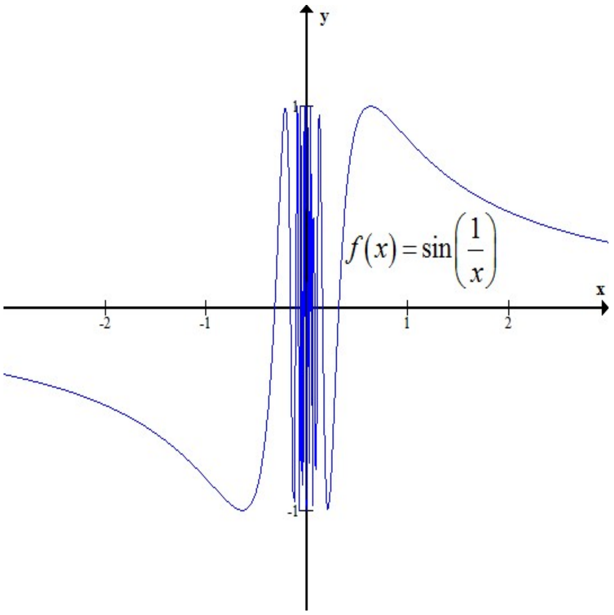

FIGURE 2.2.3. The limit of $\sin\left(\frac{1}{x}\right)$ when $x \to 0$.

Solution. We need to compute the limit

$$\begin{aligned}\lim_{t\to\infty}\frac{6t^2+5t}{(t+1)^2} &= \lim_{t\to\infty}\frac{6t^2+5t}{t^2+2t+1}\\ &= \lim_{t\to\infty}\frac{t^2}{t^2}\cdot\frac{6+\frac{5}{t}}{1+\frac{2}{t}+\frac{1}{t^2}}\\ &= \frac{6+0}{1+0+0}=6.\end{aligned}$$

EXERCISE 2.4. The organizers of Rio-2016 estimated that if the event is announced x days in advance, the revenue will be $R(x) = 400 + 120x - x^2$. The cost of advertising for x days is $C(x) = 2x^2 + 300$. What happens to the profit has $x \to \infty$? $x \to 10$?

Solution. We start with the profit function given by

$$\begin{aligned}P(x) &= R(x) - C(x)\\ &= 400 + 120x - x^2 - 2x^2 - 300\\ &= 100 + 120x - 3x^2.\end{aligned}$$

This function is a parabolic function with a global maximum point.

$$\lim_{x\to\infty} 100 + 120x - 3x^2 = \lim_{x\to\infty} x^2\left(\frac{100}{x^2} + \frac{120}{x} - 3\right) \overset{"\infty\cdot(-3)"}{=} -\infty.$$

$$\lim_{x\to 10} 100 + 120x - 3x^2 = 100 + 1200 - 3\cdot 100 = 1000.$$

EXERCISE 2.5. Compute the following limits.

(1) $\lim_{x\to 2} x^3$.
(2) $\lim_{x\to 5} x^2 - 3x + 1$.
(3) $\lim_{x\to 1} \frac{2x^2-3x+1}{x-1}$.
(4) $\lim_{x\to 2} \frac{\sqrt{4x-4}-x}{x^2-4}$.

Solution.

(1) By the arithmetic limit laws we get, $\lim_{x\to 2} x^3 = \lim_{x\to 2} x\cdot x\cdot x = 2\cdot 2\cdot 2 = 8$.
(2) By the arithmetic limit laws we get, $\lim_{x\to 5} x^2 - 3x + 1 = 25 - 15 + 1 = 11$.

(3) Note that the numerator and denominator converge to 0. Hence,

$$\begin{aligned}\lim_{x\to 1}\frac{2x^2-3x+1}{x-1} &= \lim_{x\to 1}\frac{(2x-1)(x-1)}{x-1}\\ &= \lim_{x\to 1}(2x-1)=1.\end{aligned}$$

(4) We will need some algebra in this case. Specifically, we need to multiply and divide by the conjugate of the numerator and get

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{4x-4}-x}{x^2-4} &= \lim_{x\to 2}\frac{\sqrt{4x-4}-x}{x^2-4}\cdot\frac{\sqrt{4x-4}+x}{\sqrt{4x-4}+x}\\ &= \lim_{x\to 2}\frac{4x-4-x^2}{\left(x^2-4\right)\left(\sqrt{4x-4}+x\right)}\\ &= \lim_{x\to 2}\frac{-\left(x-2\right)^2}{(x+2)(x-2)\left(\sqrt{4x-4}+x\right)}\\ &= \lim_{x\to 2}\frac{-(x-2)}{(x+2)\left(\sqrt{4x-4}+x\right)}=\frac{0}{4\left(\sqrt{4}+2\right)}=0.\end{aligned}$$

EXERCISE 2.6. Compute the following limits.

(1) $\lim_{x\to 0}\frac{|x|}{x}$.
(2) $\lim_{x\to 0^+}\frac{17-x}{x}$.
(3) $\lim_{x\to 0^+}\frac{x-1}{x}$.

Solution.

(1) This limit does not exist. To prove this, we compute the limit when $x\to 0^+$ and $x\to 0^-$.

$$\begin{aligned}\lim_{x\to 0^+}\frac{|x|}{x} &= \lim_{x\to 0^+}\frac{x}{x}=\lim_{x\to 0^+}1=1;\\ \lim_{x\to 0^-}\frac{|x|}{x} &= \lim_{x\to 0^-}\frac{-x}{x}=\lim_{x\to 0^-}-1=-1.\end{aligned}$$

(2) A direct computation shows that

$$\lim_{x\to 0^+}\frac{17-x}{x}\overset{"\frac{17}{0^+}"}{=}\infty.$$

(3) A direct computation shows that

$$\lim_{x\to 0^+}\frac{x-1}{x}\overset{"\frac{-1}{0^+}"}{=}-\infty.$$

REMARK 2.4. The value of e is defined through the following limit

$$\lim_{x\to\infty}\left(1+\frac{k}{x}\right)^x=e^k.$$

EXERCISE 2.7. Prove or disprove the following statements.

(1) If $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$, then $\lim_{x\to\infty}\left(f(x)-g(x)\right)=0$.
(2) If $\lim_{x\to\infty}\left(f(x)-g(x)\right)=0$, then $\lim_{x\to\infty}\frac{f(x)}{g(x)}=1$.

Solution.

(1) This statement is false. For instance, take $f(x)=x+1$ and $g(x)=x$. Then,

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{x+1}{x}=\lim_{x\to\infty}1+\frac{1}{x}=1;\\ \lim_{x\to\infty}\left(f(x)-g(x)\right) &= \lim_{x\to\infty}(x+1-x)=\lim_{x\to\infty}1=1.\end{aligned}$$

(2) This statement is also false. For example, take $f(x) = \frac{1}{x}$ and $g(x) = \frac{1}{x^2}$. Then,

$$\begin{aligned} \lim_{x\to\infty}(f(x) - g(x)) &= \lim_{x\to\infty}\left(\frac{1}{x} - \frac{1}{x^2}\right) = 0 + 0 = 0; \\ \lim_{x\to\infty}\frac{f(x)}{g(x)} &= \lim_{x\to\infty}\frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{x\to\infty} x = \infty. \end{aligned}$$

2.2.3. Continuous functions.

The definition of a continuous function is based on the limit of that function and the value of that function in a specific point.

DEFINITION 2.3. A function f is continuous in x_0 if the limit $\lim_{x\to x_0} f(x)$ exists, $f(x_0)$ is defined, and both values are equal, $\lim_{x\to x_0} f(x) = f(x_0)$.

In contrary to the limit's definition, a continuous function in x_0 must be defined in x_0 and the continuity depends greatly on the value $f(x_0)$.
A function does not have to be continuous. Consider for example the function

$$f(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

This function is continuous in every $x_0 \neq 0$, and it is also defined in $x_0 = 0$, however it is not continuous in $x_0 = 0$. See Figure 2.2.4 for more examples of discontinuous functions.

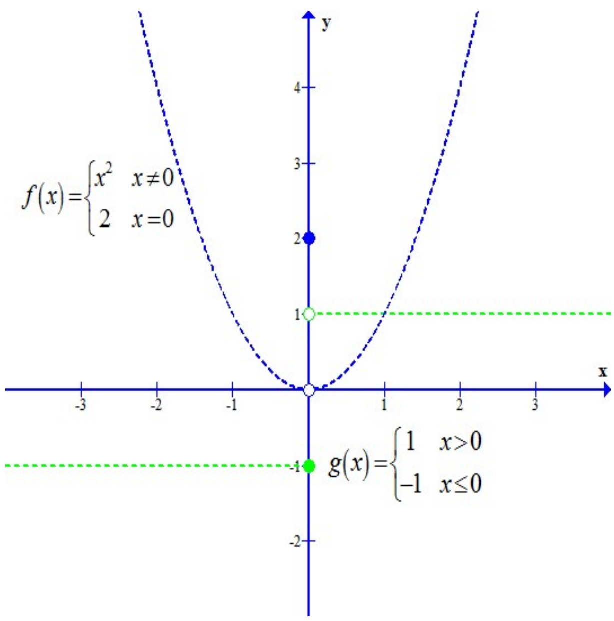

FIGURE 2.2.4. Non continuous parabolic and linear functions.

REMARK 2.5. one way of proving that a function is not continuous is by computing the limit when $x \to x_0^+$ and the limit when $x \to x_0^-$. If the limits are not the same, then the limit when $x \to x_0$ does not exist, and the function is not continuous.

EXERCISE 2.8. Find the values of A and B such that the function is continuous:

$$f(x) = \begin{cases} \frac{x^2 - x}{x-1} + A, & x < 1, \\ 3, & x = 1, \\ B \cdot \ln\left(x^2 + 2\right) - 2, & x > 1. \end{cases}$$

Solution. For every $x \neq 1$, the function is continuous, as it is either a polynomial or a logarithmic function, which are continuous. So we need to focus on $x = 1$. First, we need to make sure that the limit exists at $x = 1$.

$$\lim_{x\to 1^+} \left[B \cdot \ln\left(x^2+2\right) - 2\right] = \left[B \cdot \ln\left(1^2+2\right) - 2\right] = B\ln(3) - 2,$$

$$\lim_{x\to 1^-} \left[\frac{x^2-x}{x-1} + A\right] = \lim_{x\to 1^-} [x + A] = 1 + A.$$

Thus, for the limit to exist, we require that $A + 1 = B\ln(3) - 2$. In addition, for the function to be continuous, $f(1) = 3$ must equal the limit. To conclude,

$$A + 1 = B\ln(3) - 2 = 3 \Rightarrow A = 2,\ B = \frac{5}{\ln(3)}.$$

EXERCISE 2.9. Find the values of A, B, and C such that the function

$$f(x) = \begin{cases} x^2 - 2Ax + 5C \cdot \cos(x), & x < 0, \\ B, & x = 0, \\ \frac{\tan(2x)}{x}, & x > 0, \end{cases}$$

is continuous.

Solution. For every $x \neq 0$, the function is continuous, as it is either a polynomial or $\frac{\tan(2x)}{x}$, that are continuous. So we need to focus on $x = 0$. First, we need to make sure that the limit exists in $x = 0$.

$$\lim_{x\to 0^+} \frac{\tan(2x)}{x} = \lim_{x\to 0^+} \frac{2}{\cos(2x)} \cdot \frac{\sin(2x)}{2x} = 2 \cdot 1 = 2,$$

when we used the known limit $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$.

$$\lim_{x\to 0^-} x^2 - 2Ax + 5C \cdot \cos(x) = 0 - 0 + 5C = 5C.$$

Thus, for the limit to exist, we require that $5C = 2 \Rightarrow C = \frac{2}{5}$. In addition, for the function to be continuous, $f(0) = B$ must equal the limit, which is 2. To conclude,

$$B = 2, C = \frac{2}{5}, A \in \mathbb{R}.$$

EXERCISE 2.10. Discuss the continuity of the function

$$f(x) = \begin{cases} x^2 - 3x, & \text{if } x < 2, \\ 4 + 2x, & \text{if } x \geq 2, \end{cases}$$

on the open interval $(0, 2)$ and on the closed interval $[0, 2]$.

Solution. Since $x^2 - 3x$ is a continuous function and $f(x) = x^2 - 3x$ in $(0, 2)$, we get the f is continuous in the open interval. However, in the closed interval, the function is not continuous in $x = 2$, since $f(2) = 4 + 2 \cdot 2 = 8$ and $\lim_{x\to 2^-} x^2 - 3x = 4 - 6 = -2$.

THEOREM 2.1. **(the Mean Value Theorem)** *If a function f is continuous on the closed interval $[a, b]$, where $a < b$, and assume that there exists some value d between $f(a)$ and $f(b)$, then there exists a point $c \in (a, b)$ such that*

$$f(c) = d.$$

EXERCISE 2.11. The price p of a product is contained in the interval $[0, 1]$. The demand function $D(p)$ is given by $D(p) = 1 - p^2$. The supply function $S(p)$ is $S(p) = 0.5 + 2p - p^{1/3}$. Determine whether there exists an equilibrium price p_e where the supply meets the demand.

Solution. We need to find whether there exists an equilibrium price p_e such that the supply function equals the demand function. Specifically,

$$\begin{aligned} D(p) &= S(p) \\ D(p) - S(p) &= 0 \\ 1 - p^2 - 0.5 - 2p + p^{1/3} &= 0 \\ 0.5 - p^2 - 2p + p^{1/3} &= 0 \end{aligned}$$

We can see that we got a continuous function, therefore we can use the Mean-Value theorem.

(1) When $p = 0$, the right hand side of the previous equation equals 0.5.
(2) When $p = 1$, the right hand side of the previous equation equals $0.5 - 1 - 2 + 1 = -1.5$.

By the Mean-Value Theorem, we know that a price $p_e \in (0, 1)$ exists such that the right hand side equals 0, and the result follows. See figure 2.2.5 for a sketch of the two functions.

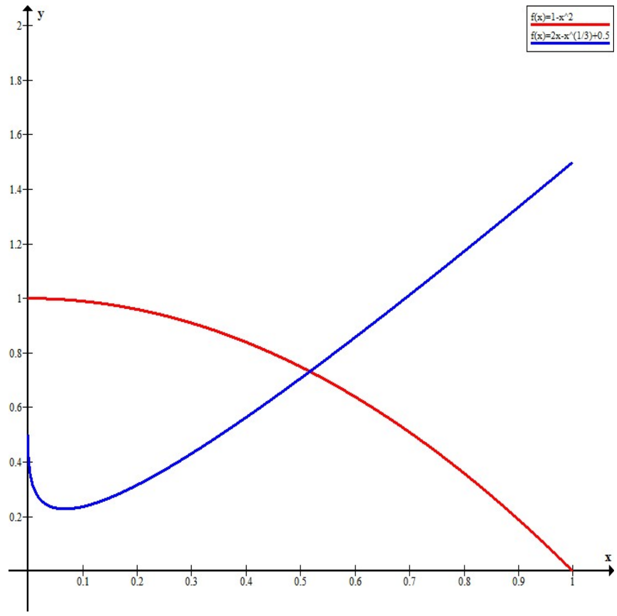

FIGURE 2.2.5. Supply meets demand from Exercise 2.11

EXERCISE 2.12. Prove that for every $0 < a < 1$, $n \in \mathbb{N}$, the equation

$$2^x = a + (2x)^n$$

has a solution $x_0 \in (0, 1)$.

Solution. Fix $0 < a < 1$, $n \in \mathbb{N}$, and define $f(x) = 2^x - a - (2x)^n$. Note that f is a continuous function. In addition, $f(0) = 1 - a > 0$ and $f(1) = 2 - a - 2^n \leq -a + 0 < 0$. By the Mean Value Theorem on $(0, 1)$, there exists a point $c \in (0, 1)$ such that $f(c) = 0$. Hence,

$$\begin{aligned} 2^c - a - (2c)^n &= 0 \\ 2^c &= a + (2c)^n, \end{aligned}$$

as needed.

2.3. Derivatives

Similarly to the concept of continuity, the idea behind the derivative of a function is intuitively simple, but the definition tends to be complex. The basic idea behind the derivative of a function is to measure the change in the values of that function. Specifically, the derivative is a function that states how fast or slow a function increases or decreases. Nevertheless, the main problem of measuring the derivative at a certain point is not that simple.

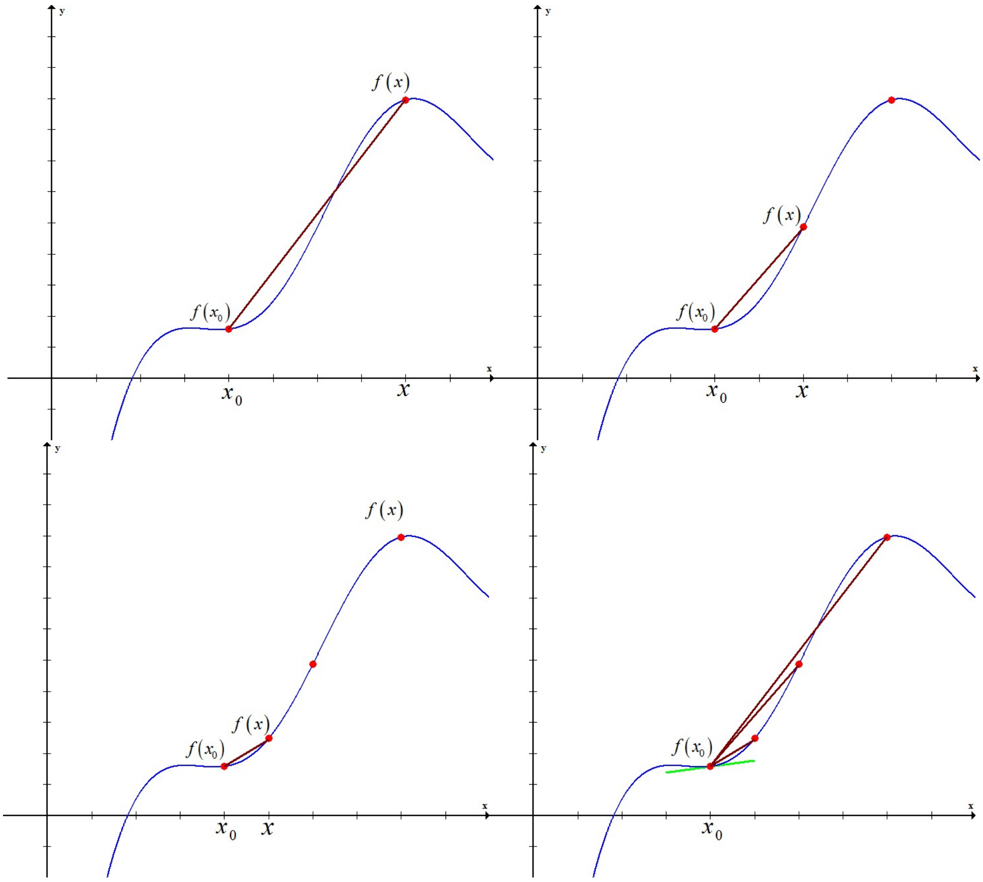

FIGURE 2.3.1. The derivative of a function f at a point x_0.

The derivative $f'(x_0)$ of the function $f(x)$ in x_0 is defined through the inclination\slope of the function in that point. To define this value correctly, we consider the value of the function in two different points x_0 and x_0. We draw a straight line between $f(x)$ and $f(x_0)$ and calculate the angle of that line w.r.t. the x axis. Then, we take xs that are closer and closer to x_0 and recalculate that value. by taking the limit, we get $f'(x_0)$. Formally,

$$\begin{aligned} f'(x_0) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \\ &\overset{h=x-x_0}{=} \lim_{h\to 0} \frac{f(x+h)-f(x_0)}{h}. \end{aligned}$$

We can try to find the derivative in every point and by doing so, we get a new function $f'(x)$, which represents the slope of f at any point x (as long as the limit above exists). It terms of notations, we sometime use $\frac{d}{dx}f(x)$ to denote the derivative of the function $f(x)$.

REMARK 2.6. If a function is differentiable on an interval, it is also continuous on that interval.

EXERCISE 2.13. Find the derivative of the function$f(x) = 16x^2$ using only the definition.

Solution. Using the derivative's definition yields

$$\begin{aligned} \lim_{h\to 0} \frac{16(x+h)^2 - 16x^2}{h} &= \lim_{h\to 0} \frac{16\left(x^2+2xh+h^2\right) - 16x^2}{h} \\ &= \lim_{h\to 0} \frac{32xh+16h^2}{h} \\ &= \lim_{h\to 0} 32x + 16h = 32x. \end{aligned}$$

EXERCISE 2.14. Prove that $f(x) = |x - 13|$ is differentiable anywhere but $x_0 = 13$.

Solution. We compute the derivative using the basic definition when $x_0 > 13$, $x_0 < 13$, and when $x_0 = 13$.

- If $x_0 > 13$, then
$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{|x_0+h-13|-|x_0-13|}{h} \\ &= \lim_{h\to 0}\frac{x_0+h-13-x_0+13}{h} \\ &= \lim_{h\to 0}\frac{h}{h} = 1. \end{aligned}$$
- If $x_0 < 13$, then
$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{|x_0+h-13|-|x_0-13|}{h} \\ &= \lim_{h\to 0}\frac{-(x_0+h-13)+(x_0+13)}{h} \\ &= \lim_{h\to 0}\frac{-h}{h} = -1. \end{aligned}$$
- If $x_0 = 13$, then
$$\begin{aligned} f'(x_0) &= \lim_{h\to 0}\frac{|13+h-13|-|13-13|}{h} \\ &= \lim_{h\to 0}\frac{|h|}{h} = \begin{cases} +1, & h\to 0^+, \\ -1, & h\to 0^-, \end{cases} \end{aligned}$$
and the limit does not exist, hence the derivative does not exist.

2.3.1. Standard economic uses for derivatives.

- **Finding extreme points** (specified later on).
- **Increasing or decreasing functions**. The derivatives specifies whether a function is increasing or decreasing. For example, if the marginal cost is positive, we know that an increase in production will result in an increase in cost.
- **Marginals estimations.** In economics we usually look at the marginal cost or marginal revenue, which means the change in cost or the change in revenue when a small change in the production is made. Consider, for example, the cost function $C(x)$ which defines the cost for producing x units. The marginal cost (per unit) for producing two more units is
$$\frac{C(x+2)-C(x)}{2}.$$
Now we can take a smaller and smaller increase and get the marginal cost as the limit
$$\lim_{h\to 0}\frac{C(x+h)-C(x)}{h}.$$

2.3.2. Derivatives of basic functions.

Since we would not want to make the same computation for every function and for every $x \in \mathbb{R}$, we can formulate the derivatives of certain basic functions. The following list contains the derivatives of functions we use regularly (any other derivative could be computed directly according to the original definition):

The function	The derivative	The function	The derivative
$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
c	0	a^x	$a^x \ln(a)$
$x^n \quad, n \neq 0$	nx^{n-1}	$\log_a(x)$	$\frac{1}{x\ln(a)}$
$\sin(x)$	$\cos(x)$	$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos(x)$	$-\sin(x)$	$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan(x)$	$\frac{1}{\cos^2(x)}$	$\arctan(x)$	$\frac{1}{1+x^2}$

2.3.3. Rules of differentiation.

There are a few elementary rules of differentiation. Let f, g be two differentiable functions, then:

(1) $(f(x) \pm g(x))' = f'(x) \pm g'(x)$.
(2) $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$.
(3) $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$, when $g(x) \neq 0$.
(4) An inverse function $f^{-1}(x)$ is differentiable if and only if $f'\left(f^{-1}(x)\right)$ exists and does not equal 0. The derivative is given by

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

(5) Another important rule of differentiation is *the chain rule*. Define $h(x) = f(g(x))$ and assume that $g(x)$ is differentiable in x and f is differentiable in $g(x)$, then $h(x)$ is also differentiable in x and $h'(x) = f'(g(x))g'(x)$.

These rules are very helpful when trying to compute the derivative of non-basic functions. For example,

$$\begin{aligned} f(x) &= \sqrt{x},\ g(x) = x^2+1 \ \Rightarrow h(x) = f\left(x^2+1\right) = \sqrt{x^2+1} \\ h'(x) &= \frac{1}{2\sqrt{x^2+1}} \cdot 2x. \end{aligned}$$

EXERCISE 2.15. The manager of an appliance manufacturing firm determines that when blenders are priced at p dollars apiece, the number sold each month can be modeled by $D(p) = \frac{8000}{p}$. The manager estimates that t months from now, the unit price of the blenders will be $p(t) = 0.06t^{1.5} + 22.5$ dollars. At what rate will the monthly demand for blenders $D(p)$ be changing 25 months from now? Will it be increasing or decreasing at this time?

Solution. We want to find how the demand D changes as a function of the time. Using the chain rule,

$$\begin{aligned} \frac{d}{dt}\left[D(p(t))\right] &= \frac{dD(p)}{dp} \cdot \frac{dp(t)}{dt} \\ &= -\frac{8000}{p^2} \cdot \frac{6}{100} \cdot \frac{3}{2}t^{0.5}. \end{aligned}$$

Thus, when $t = 25$, we get $p(25) = 0.06 \cdot (25)^{1.5} + 22.5 = 30$, and

$$\frac{d}{dt}\left[D(p(t))\right]_{t=25} = -\frac{720}{(30)^2} \cdot 25^{0.5} = -4.$$

That is, the demand for blenders will be decreasing at a rate of 4 units per month.

2.4. Extreme points

In many economic model we try to maximize or minimize the payoff, utility, or some kind of a production function. In order to do so, we need to thoroughly understand the methods of optimizing any given function. For that purpose, we study extreme points of functions. We start with the basic definitions.

DEFINITION 2.4. Let $f : D \to \mathbb{R}$ be a one-variable function with domain $D \subseteq \mathbb{R}$. The point $x_0 \in D$ is a global maximum (minimum) if $f(x_0) \geq f(x)$ $(f(x_0) \leq f(x))$ for every $x \in D$.

In other words, a global maximum (or, a global minimum), is a point x_0 that maximizes (minimizes) the value of f with respect to all the other point in D.

DEFINITION 2.5. Let $f : D \to \mathbb{R}$ be a one-variable function with domain $D \subseteq \mathbb{R}$. The point $x_0 \in D$ is a local maximum (minimum) if there exists an open interval I containing x_0 such that $f(x_0) \geq f(x)$ $(f(x_0) \leq f(x))$ for every $x \in I$.

In general, finding the extreme points of a function is not easy. Therefore, we have a few theorems that assist with this problem. The first theorem is the Bolzano–Weierstrass theorem that gives a basic condition for extreme points to exist.

THEOREM 2.2. **(Bolzano–Weierstrass)** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$. Then f must attain a maximum and a minimum, each at least once.*

That is, there exist numbers $c, d \in [a, b]$ such that $f(c) \geq f(x) \geq f(d) \quad \text{for all } x \in [a, b]$. The theorem is based solely on continuity of f. The next theorem, Fermat's Theorem, relates to cases where an inner extreme point of a differentiable function has a very unique property such that the derivative is zero.

THEOREM 2.3. **(Fermat)** *Let $f : (a, b) \to \mathbb{R}$ be a function and suppose that $x_0 \in (a, b)$ is a local extreme point of f. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Note that the theorem does not mean that every point where the derivative equals zero is an extreme point. This is not true in general. The theorem states that in cases that f is differentiable, the derivative in the extreme points is zero. Thus, in order to find extreme point, one need to solve the equation $f'(x) = 0$ and all the points that solve this equation are suspected to be extreme points.
Theorem 2.3 and Theorem 2.2 make the problem of finding extreme point easier, using the following stages:

(1) Find all the point where $f'(x) = 0$ or $f'(x)$ is not defined. These points are called *critical points* and are suspected to be extreme points.
(2) Compute the values $f(x)$ for each critical point.
(3) Compare the values and determine the properties of each point.

EXERCISE 2.16. A company produces a product at a cost of 5\$ each. The company assumes that if the price for each product is x, then $15 - x$ products will be sold. What is the company's profit function? What should it charge in order to maximize its profit?

Solution. In this exercise x is the market price, which is a choice variable for the firm. The profit function of the firm is

$$\begin{aligned} P(x) &= x(15 - x) - 5(15 - x) \\ &= (15 - x)(x - 5) \\ &= -x^2 + 20x - 75. \end{aligned}$$

This function is concave, and its first derivative is $P'(x) = -2x + 20$. The function reaches its maximal value at $x = 10$.

EXERCISE 2.17. A company produces a product at a cost of 5\$ each. The current price is 10\$ apiece and 10 products are sold each day. The company realizes that each dollar decrease in the price, will increase the amount of products sold by 1 product a day. Write the demand and profit functions and find the price that maximizes the profit.

Solution. From the information given, the demand function $D(p)$ (as a function of the price) must be computed. The function is linear, and the slope is -1. It goes through the point $(10, 10)$, so the function must be

$$\begin{aligned} D(p) &= -p+b \\ D(10) &= -10+b=10 \\ \Rightarrow b &= 20, \\ D(p) &= -p+20. \end{aligned}$$

Then the profit function (as a function of price) must be $\pi(p) = (p-5)(20-p)$. A direct computation shows that $\pi'(p) = 25-2p$, so the profit is maximized at $p = 12.5$.

EXERCISE 2.18. To produce x units of a particular commodity, a monopolist has a total cost of

$$C(x) = 2x^2+3x+5,$$

and total revenue of $R(x) = xp(x)$, where $p(x) = 5-2x$ is the price at which the x units will be sold. Find the profit function $P(x)$. For what level of production is profit maximized?

Solution. The profit function is

$$\begin{aligned} P(x) &= R(x)-C(x) \\ &= xp(x)-2x^2-3x-5 \\ &= 5x-2x^2-2x^2-3x-5 \\ &= -4x^2+2x-5. \end{aligned}$$

Taking the first-order condition yields

$$\begin{aligned} P'(x) &= -8x+2=0 \\ \Rightarrow \quad x = &\ 0.25. \end{aligned}$$

Thus, the profit is maximized when producing just a single unit.

EXERCISE 2.19. When interest rates are low, many homeowners take the opportunity to refinance their mortgages. As rates start to rise, there is often a flurry of activity as latecomers rush in to refinance while they still can do so profitably. Eventually, however, rates reach a level where refinancing begins to wane. Suppose in a certain community, there will be $M(r)$ thousand refinanced mortgages when the 30-year fixed mortgage rate is $r\%$, where

$$M(r) = \frac{1+0.05r}{1+0.004r^2}, \quad \text{for } 1 \le r \le 8.$$

(1) For what values of r is $M(r)$ increasing?
(2) For what interest rate r is the number of refinanced mortgages maximized? What is this maximum number?

Solution.

(1) Note that we have a continuous function in a closed interval, thus a maxima and a minima exist. Taking the first-order condition, we get

$$\begin{aligned} M'(r) &= \frac{0.05\left(1+0.004r^2\right)-0.008r\,(1+0.05r)}{\left(1+0.004r^2\right)^2} \\ &= \frac{0.05+0.0002r^2-0.008r-0.0004r^2}{\left(1+0.004r^2\right)^2} \\ &= \frac{-2r^2-80r+500}{10000\cdot\left(1+0.004r^2\right)^2}. \end{aligned}$$

Thus, we see that M increases according to the sign of $-2r^2 - 80r + 500$. Finding the solution for the equation $-2r^2 - 80r + 500 = 0$,

$$\begin{aligned} r_{1,2} &= \frac{80 \pm \sqrt{6400 + 4000}}{-4} \\ &= -20 \pm \sqrt{650} \\ &= -20 \pm 25.5 \\ r_1 &= 5.5 \\ r_2 &= -45.5. \end{aligned}$$

Since the function we are analyzing is a parabolic function with a global maxima, we know that $M(r)$is increasing when $1 \leq r \leq 5.5$.

(2) The function is maximized at the very end point it stops increasing. In other words, when $r = 1$ we have a minima ($M(1) = \frac{1.05}{1.004} \approx 1.05$), whereas $r = 5.5$ is a maxima, therefore M is maximized when $r = 5.5$. The maximum value is $M(5.5) = \frac{1+0.05 \cdot 5.5}{1+0.004 \cdot 5.5^2} \approx 1.14$. Note that the value at the other end point $r = 8$ is $M(8) \approx 1.1$.

EXERCISE 2.20. Give an economic interpretation of the derivatives of the following functions:

(1) $F(q)$ is the revenue from producing q units of output;
(2) $G(x)$ is the cost of purchasing x unit of some commodity.
(3) $H(p)$ is the amount of commodity consumed when its price is p.
(4) $C(Y)$ is the total consumption when national income is Y.
(5) $S(Y)$ is the total savings when national income is Y.

Solution.

(1) The derivative is the **marginal revenue**, that is, the rate at which revenue increases with output.
(2) The derivative is the **marginal cost**, that is, the rate at which the cost of purchasing x units increases with x.
(3) The derivative is the rate at which demand increases with price.
(4) The derivative is the **marginal propensity to consume**, that is, the rate at which aggregate consumption increases with national income.
(5) The derivative is the **marginal propensity to save**, that is, the rate at which aggregate savings increases with national income.

EXERCISE 2.21. A manufacturer estimates that when x units of a particular commodity are produced, the total cost will be $C(x) = \frac{1}{8}x^2 + 3x + 98$ dollars, and furthermore, that all x units will be sold when the price is $p(x) = \frac{1}{3}(75 - x)$ dollars per unit.

(1) Use marginal cost to estimate the cost of producing the ninth unit. What is the actual cost of producing the ninth unit?
(2) Find the marginal profit.
(3) Use marginal revenue to estimate the revenue derived from the sale of the ninth unit. What is the actual revenue derived from the sale of the ninth unit?

Solution.

(1) The marginal cost is $C'(x) = \frac{1}{4}x + 3$. The cost of producing the ninth unit is the change in cost as x increases from 8 to 9 and can be estimated by the marginal cost

$$C'(8) = \frac{1}{4} \cdot 8 + 3 = \$5.$$

The actual cost of producing the ninth unit is

$$C(9) - C(8) = \$5.13.$$

(2) The revenue is given by $R(x) = xp(x) = 25x - \frac{1}{3}x^2$. The profit is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 25x - \frac{1}{3}x^2 - \frac{1}{8}x^2 - 3x - 98 \\ &= 22x - 98 - \frac{11}{24}x^2, \\ P'(x) &= 22 - \frac{11}{12}x. \end{aligned}$$

(3) The revenue obtained from the sale of the ninth unit is approximated by the marginal revenue

$$R'(8) = 25 - \frac{2}{3} \cdot 8 = \$19.67.$$

The actual revenue obtained from the sale of the ninth unit is

$$R(9) - R(8) = \$19.33.$$

EXERCISE 2.22. Answer the following questions:

(1) Is the function $f(x) = 2x^3 - 12x^2$ increasing or decreasing in $x = 3$?
(2) Is the function $f(x) = \ln(x)$ increasing or decreasing in $x = 13$?
(3) Is the function $f(x) = e^{-x}x^{1.5}$ increasing or decreasing in $x = 4$?
(4) Is the function $f(x) = \frac{4x-1}{x+2}$ increasing or decreasing in $x = 2$?
(5) Is the function $f(x) = \frac{3x-2}{4x+x^2}$ increasing or decreasing in $x = -1$?
(6) Is the function $f(x) = \frac{1}{\ln(x)}$ increasing or decreasing in $x = e$?
(7) Is the function $f(x) = 5x^2 + 16x - 12$ increasing or decreasing in $x = -6$?

Solution.

(1) The derivative is $f'(x) = 6x^2 - 24x$. Thus, $f'(3) = 54 - 72 = -18 < 0$ and the function is decreasing.
(2) The derivative is $f'(13) = \frac{1}{13} > 0$, and the function is increasing.
(3) The derivative is $f'(4) = -5e^{-4} < 0$, and the function is decreasing.
(4) The derivative is $f'(2) = \frac{9}{16} > 0$, and the function is increasing.
(5) The derivative is $f'(-1) = \frac{1}{9} > 0$, and the function is increasing.
(6) The derivative is $f'(e) = -\frac{1}{e} < 0$, and the function is decreasing.
(7) The derivative is $f'(-6) = -44 < 0$, and the function is decreasing.

EXERCISE 2.23. A firm can use its manufacturing facility to make either tables or chairs. Both require labor only. The production function for tables is

$$A = 20L^{0.5},$$

and the production function for chairs is

$$B = 30L.$$

The wage rate is \$11 per unit of time, and the prices of tables and chairs are \$9 and \$3 per unit, respectively. The manufacturing facility can accommodate 60 workers and no more. How much of each product should the firm produce per unit of time?

Solution. Assume that the firm devotes L units of labor for tables production and $60 - L$ to chairs production. The profit function is

$$\begin{aligned} \pi(L) &= 9 \cdot 20L^{0.5} + 3 \cdot 30(60 - L) - 11 \cdot 60 \\ &= 180L^{0.5} - 90L + 79 \cdot 60. \end{aligned}$$

The FOC is $\pi'(L) = 90\frac{1}{\sqrt{L}} - 90 = 0$, which means that $L = 1$. Thus, the firm produces 20 tables and 1770 chairs.

2.4.1. Derivatives of second and higher order.

In the same manner that a function could be differentiable, its derivative (which is also a function) could be differentiable as well. In this case, we say that f is twice differentiable and f'' is the second derivative of f. The same is also true for the second derivative and so on. When a function is k times differentiable and the k^{th} derivative is continuous, we say that the original function is k times continuously differentiable and denote this set of functions by C^k. The k derivative of f is denoted by $f^{(k)}(x)$.

2.5. Convex & concave functions

Convexity and concavity are important properties of functions, specifically in Economics. These attributes help us with determining the number of extreme points (or equilibrium) and their nature. But first, let us define the two properties.

DEFINITION 2.6. **(Convex and Concave functions)** A function f is called *convex* on an interval I if and only if $f((1-t)a+tb) \leq (1-t)f(a)+tf(b)$ for every $a, b \in I$ and all $t \in [0,1]$.
A function f is called *concave* on an interval I if and only if $f((1-t)a+tb) \geq (1-t)f(a)+tf(b)$ for every $a, b \in I$ and all $t \in [0,1]$.

For example, $f(x) = x^2$ and $f(x) = 1/x$ are convex functions, whereas $f(x) = \sqrt{x}$ and $f(x) = \ln(x)$ are concave functions. When function change from concavity to convexity, or otherwise, that point is called an *inflection point*. As previously stated, convexity or concavity are very helpful properties when trying to determine whether an extreme point is either a unique maximum, or a unique minimum.

PROPOSITION 2.1. *Let f be a function with an extreme point x_0. If f is concave (convex), then x_0 is the unique maxima (minima, resp.) of f.*

One way to characterize functions by these properties is through the Second-Order Condition (SOC), relating to the second-order derivative.

PROPOSITION 2.2. *When a function is twice differentiable, the second derivative determines whether the function is convex or concave. Specifically, if $f''(x) \geq 0$ ($f''(x) \leq 0$), then the function is convex (concave).*

EXERCISE 2.24. Optimize the following functions and tell whether the optimum is a local maximum or a local minimum

(1) $f(x) = -4x^2 + 10x$.
(2) $f(x) = 120x^{0.7} - 6x$.
(3) $f(x) = 4x - 3\ln(x)$.

Solution.

(1) The FOC is $f'(x) = -8x + 10 = 0$. So, $x* = \frac{5}{4}$. The SOC is $f''\left(\frac{5}{4}\right) = -8 < 0$, and this is a local maximum.
(2) The FOC is $f'(x) = \frac{84}{x^{0.3}} - 6 = 0$. So, $x* = 14^{1/0.3}$. The SOC is $f''\left(14^{1/0.3}\right) < 0$, and this is a local maximum.
(3) The FOC is $f'(x) = 4 - \frac{3}{x} = 0$. So, $x* = \frac{3}{4}$. The SOC is $f''\left(\frac{3}{4}\right) > 0$, and this is a local minimum.

EXERCISE 2.25. During a recession, Congress decides to stimulate the economy by providing funds to hire unemployed workers for government projects. Suppose that t years after the stimulus program begins, there are $N(t)$ thousand people unemployed, where

$$N(t) = \frac{1}{3}t^3 - 3t^2 + 37 \quad \text{where } 0 \leq t \leq 10.$$

(1) What is the maximum number of unemployed workers? When does the maximum level of unemployment occur?
(2) In order to avoid overstimulating the economy (and inducing inflation), a decision is made to terminate the stimulus program as soon as the decrease in the unemployment rate begins to weaken. When does this occur? At this time, how many people are unemployed?

Solution. To solve this problem we need to find the first and second derivatives of N.

$$\begin{aligned} N'(t) &= t^2 - 6t, \\ N''(t) &= 2t - 6. \end{aligned}$$

(1) Note that $N(t) \to \infty$ as $t \to \infty$, and the function is continuous. Thus, there is a global maxima and a global minima. We need to identify critical point first. We have two simple solutions: $t_1 = 0$ and $t_2 = 6$. In addition, we can see that the function is decreasing when $0 \le t \le 6$, and increasing otherwise. Hence,

$$\begin{aligned} f(0) &= 37, \\ f(6) &= 72 - 108 + 37 = 1, \\ f(10) &= 333\frac{1}{3} - 300 + 37 = 70\frac{1}{3}. \end{aligned}$$

Thus, the maximal level of unemployment occurs when $t = 10$, and it is 70.33%.
(2) Now, we need to analyze the change in the unemployment rate. We sew that there is a decrease in unemployment when $0 \le t \le 6$. But we are asked when the decrease starts to weaken, that is when $N'(t)$ starts to increase. By analyzing the second derivative we see that $t = 3$ is an inflection point of N and the derivative of N starts to increase. At that stage, unemployment is $N(3) = 9 - 27 + 37 = 19\%$.

EXERCISE 2.26. Determine when the following functions are convex or concave.

(1)
$$f(x) = x^2 + 4x - 17.$$
(2)
$$f(x) = \ln(x).$$
(3)
$$f(x) = \frac{1}{x}.$$
(4)
$$f(x) = \begin{cases} x + 2, & x < -1, \\ \frac{1}{4}x + \frac{5}{4}, & x \in [-1, 1], \\ \frac{7}{2} - 2x & x > 1. \end{cases}$$

Solution.

(1) We know this is a parabolic function. It is differentiable infinitely many times. Thus

$$\begin{aligned} f'(x) &= 2x + 4, \\ f''(x) &= 2 > 0 \end{aligned}$$

and the function is convex.
(2) The function is defined for every $x > 0$. It is differentiable, thus

$$\begin{aligned} f'(x) &= \frac{1}{x}, \\ f''(x) &= -\frac{1}{x^2} < 0, \end{aligned}$$

and the function is concave for every $x > 0$.

(3) The function is defined for every $x \neq 0$. It is differentiable, thus

$$\begin{aligned} f'(x) &= -\frac{1}{x^2}, \\ f''(x) &= \frac{2}{x^3}. \end{aligned}$$

This means that it is convex for every $x > 0$ and the function is concave for every $x < 0$.

(4) Note that the function is defined by three linear functions. Although the derivatives do not exist when $x = -1, 1$, we can use the basic definition of concave and convex functions and see that this function is concave. Note that a linear function is both concave and convex, so when x is restricted to a specific linear function ($x < 1$, or $x \in [-1, 1]$, or $x > 1$), then the function is both convex and concave.

2.6. Derivatives Theorems

There are several major theorems concerning derivatives. In many cases, we use them, even without knowing, because they are so intuitive.

THEOREM 2.4. **(Rolle)** *If a real-valued function f is continuous on a closed interval $[a, b]$, differentiable on the open interval (a, b), and $f(a) = f(b)$, then there exists at least one c in the open interval (a, b) such that $f'(c) = 0$.*

In simple terms, Rolle's Theorem states that when a function is differentiable and when it reaches the same value at twice, say in $x = a$ and in $x = b$, then there exists a point $c \in (a, b)$ such that the derivative $f'(c)$ is 0.

THEOREM 2.5. **(Lagrange)** *If a function f is continuous on the closed interval $[a, b]$, where $a < b$, and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Lagrange's Theorem was generalized by Cauchy in the following manner.

THEOREM 2.6. **(Cauchy's Theorem)** *If functions f and g are both continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists some $c \in (a, b)$, such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$

Of course, if $g(a) \neq g(b)$ and if $g'(c) \neq 0$, this is equivalent to

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

EXERCISE 2.27. An airline company estimates that when a round-trip ticket between Los Angeles and San Francisco costs x dollars, the daily supply for tickets would be $S(x) = x^5$. In addition, the daily demand for tickets as a function of the price is $D(x) = n - mx$ where $m > 0$. Prove there exists a unique equilibrium price.

PROOF. An equilibrium is reached whenever supply meets demand. Thus, $x^5 = n - mx$. Define $f(x) = x^5 + mx - n$. Note that

$$\begin{aligned} \lim_{x\to\infty} x^5 + mx - n &= \lim_{x\to\infty} x^5 \left(1 + \frac{m}{x^4} - \frac{n}{x^5}\right) = \infty; \\ \lim_{x\to-\infty} x^5 + mx - n &= \lim_{x\to-\infty} x^5 \left(1 + \frac{m}{x^4} - \frac{n}{x^5}\right) = -\infty, \end{aligned}$$

and by the fact that every polynomial is a continuous function it follows from the Mean Value Theorem, Theorem 2.1, that there exists a point d such that $f(d) = 0$. Now we can prove, by contradiction, that this value is unique. Assume, to the contrary, there exists $d_1 \neq d$ such that $f(d_1) = 0$. By Rolle's

Theorem, Theorem 2.4, we know that there exists c between d and d_1 such that $f'(c) = 0$. Note that $f'(x) = 5x^4 + m$. Since $m > 0$, it follows that $f'(x) > 0$ for every $x \in \mathbb{R}$. Contradiction. $\square$

EXERCISE 2.28. An airline company estimates that when a round-trip ticket between Los Angeles and San Francisco costs x dollars, the daily supply for tickets would be $S(x) = x^{10} + 2x^2$. In addition, the daily demand for tickets as a function of the price is $D(x) = 18 - x^4$. Prove there exists exactly two equilibrium prices.

PROOF. An equilibrium is reached whenever supply meets demand. Thus, $x^{10} + 2x^2 = 18 - x^4$. Define $f(x) = x^{10} + x^4 + 2x^2 - 18$. Note that

$$\begin{aligned}\lim_{x\to\infty} x^{10} + x^4 + 2x^2 - 18 &= \lim_{x\to\infty} x^{10}\left(1 + \frac{1}{x^6} + \frac{2}{x^8} - \frac{18}{x^{10}}\right) = \infty;\\ \lim_{x\to-\infty} x^{10} + x^4 + 2x^2 - 18 &= \lim_{x\to-\infty} x^{10}\left(1 + \frac{1}{x^6} + \frac{2}{x^8} - \frac{18}{x^{10}}\right) = \infty,\end{aligned}$$

and note that $f(0) = -18$. This means that there is at least two points $d_1 < 0 < d_2$ such that $f(d_1) = f(d_2) = 0$. Now, assume, to the contrary, that there are at least three different solution. Denote them by d_1, d_2, and d_3. Without loss of generality, assume that $d_1 < d_2 < d_3$. By Rolle's Theorem, Theorem 2.4, it follows that there exists c_1, c_2 such that $d_1 < c_1 < d_2 < c_2 < d_3$ and $f'(c_1) = f'(c_2) = 0$. Let us compute $f'(x)$ explicitly. $f'(x) = 10x^9 + 4x^3 + 4x$ and we can use Theorem 2.4 again and get that there exists a point c_3 such that $c_1 < c_3 < c_2$ and $f''(c_3) = 0$. However, $f''(x) = 90x^8 + 12x^2 + 4 > 0$ for every $x \in \mathbb{R}$. Contradiction. $\square$

2.6.1. L'Hôpital's rule.

There are limits that are complicated to compute. Usually, limits of fractions when both the denominator and the numerator converge to 0 or $\pm\infty$. In these case, we can use L'Hôpital's rule:

THEOREM 2.7. **(L'Hôpital's rule)** *Consider two functions f and g which are differentiable on an open interval I except possibly at a point c contained in I. If*

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0 \text{ or } \pm\infty,$$

and $\lim_{x\to c}\frac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x in I with $x \neq c$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

The theorem also holds when $c = \pm\infty$ and $I = \mathbb{R}$.

EXERCISE 2.29. Compute the following limits:

(1) $\lim_{x\to\infty}(x - \sin(x))$.
(2) $\lim_{x\to\infty}\frac{2^x}{3^x+2^x}$.
(3) $\lim_{x\to 0}\frac{\sin(x)}{x}$.
(4) $\lim_{x\to 0}\frac{\sin(x^3)}{x}$.
(5) $\lim_{x\to 0}\frac{1-\cos(x)}{x^2}$.
(6) $\lim_{x\to 0}\frac{\ln(1+x)}{x}$.
(7) $\lim_{x\to 0}\frac{a^x-1}{x}$.
(8) $\lim_{x\to 0} x^x$.
(9) $\lim_{x\to\infty} x^{1/x}$.

Solution.

(1) We can use the fact that $\sin(x)$ is bounded and get

$$\lim_{x\to\infty}(x - \sin(x)) = \lim_{x\to\infty} x\left(1 - \frac{\sin(x)}{x}\right) = \text{"}\infty\cdot 1\text{"} = \infty.$$

(2) Dividing the numerator and the denominator by 3^x yields

$$\lim_{x\to\infty}\frac{2^x}{3^x+2^x}=\lim_{x\to\infty}\frac{\left(\frac{2}{3}\right)^x}{1+\left(\frac{2}{3}\right)^x}=\frac{0}{0+1}=0.$$

(3) Using L'Hôpital's rule yields

$$\lim_{x\to 0}\frac{\sin(x)}{x}\overset{"\frac{0}{0}"}{\underset{L}{=}}\lim_{x\to 0}\frac{\cos(x)}{1}=1.$$

(4) Using the previous limits, we get

$$\lim_{x\to 0}\frac{\sin\left(x^3\right)}{x}=\lim_{x\to 0}x^2\frac{\sin\left(x^3\right)}{x^3}=0\cdot 1=0.$$

(5) Using L'Hôpital's rule and a previous limit,

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}\overset{"\frac{0}{0}"}{\underset{L}{=}}\lim_{x\to 0}\frac{\sin(x)}{2x}=\frac{1}{2}.$$

(6) By L'Hôpital's rule, we see that

$$\lim_{x\to 0}\frac{\ln(1+x)}{x}\overset{"\frac{0}{0}"}{\underset{L}{=}}\lim_{x\to 0}\frac{\frac{1}{1+x}}{1}=1.$$

(7) By L'Hôpital's rule, we see that

$$\lim_{x\to 0}\frac{a^x-1}{x}\overset{"\frac{0}{0}"}{\underset{L}{=}}\lim_{x\to 0}\frac{a^x\ln(a)}{1}=\ln(a).$$

(8) Using the laws of exponential and logarithmic functions

$$\begin{aligned}\lim_{x\to 0^+}x^x &= \lim_{x\to 0^+}e^{\ln\left(x^x\right)}\\ &= \lim_{x\to 0^+}e^{x\ln(x)}.\end{aligned}$$

Now we can compute directly the limit $\lim_{x\to 0}x\ln(x)$ and then use the continuity of e^x.

$$\begin{aligned}\lim_{x\to 0^+}x\ln(x) &= \lim_{x\to 0^+}\frac{\ln(x)}{\frac{1}{x}}\\ &\overset{"\frac{0}{0}"}{\underset{L}{=}} \lim_{x\to 0^+}\frac{\frac{1}{z}}{-\frac{1}{x^2}}\\ &= \lim_{x\to 0^+}-x=0,\end{aligned}$$

and by continuity we get that $\lim_{x\to 0^+}x^x=\lim_{x\to 0^+}e^{x\ln(x)}=e^0=1$.

(9) Similarly to the previous exercise,

$$\begin{aligned}\lim_{x\to\infty}x^{\frac{1}{x}} &= \lim_{x\to\infty}e^{\ln\left(x^{1/x}\right)}\\ &= \lim_{x\to\infty}e^{\frac{\ln(x)}{x}}.\end{aligned}$$

Now we can compute directly the limit $\lim_{x\to\infty}\frac{\ln(x)}{x}$ and then use the continuity of e^x.

$$\lim_{x\to\infty}\frac{\ln(x)}{x}\overset{"\frac{\infty}{\infty}"}{\underset{L}{=}}\lim_{x\to\infty}\frac{\frac{1}{x}}{1}=0,$$

and by continuity we get that $\lim_{x\to\infty}x^{\frac{1}{x}}=e^0=1$.

EXERCISE 2.30. Compute the following limits:

- . $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right)=?$
- . $\lim_{x\to 0}\frac{e^{2x}-1}{x}=?$
- . $\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin(x)}=?$
- . $\lim_{x\to\infty}\frac{2^x}{x}=?$
- . $\lim_{x\to 0}\frac{e^{x^2}-1-x^2}{x^4}=?$
- . $\lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{\sin^4(x)}=?$

Solution.

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin(x)}\right) &= \lim_{x\to 0}\left(\frac{\sin(x)-x}{x\sin(x)}\right) \overset{"\frac{0}{0}"}{=} \\ &= \lim_{x\to 0}\left(\frac{\cos(x)-1}{\sin(x)+x\cos(x)}\right) \overset{"\frac{0}{0}"}{=} \\ &= \lim_{x\to 0}\left(\frac{-\sin(x)}{\cos(x)+\cos(x)-x\sin(x)}\right) = \\ &= \frac{-0}{1+1-0}=0\end{aligned}$$

$$.\lim_{x\to 0}\frac{e^{2x}-1}{x} \overset{"\frac{0}{0}"}{=} \lim_{x\to 0}\frac{2e^{2x}}{1}=2$$

$$.\lim_{x\to 0}\frac{e^{x}-e^{-x}}{\sin(x)} \overset{"\frac{0}{0}"}{=} \lim_{x\to 0}\frac{e^{x}+e^{-x}}{\cos(x)}=\frac{1+1}{1}=2$$

$$.\lim_{x\to \infty}\frac{2^{x}}{x} \overset{"\frac{\infty}{\infty}"}{=} \lim_{x\to \infty}\frac{2^{x}\ln(2)}{1}=\infty$$

$$\begin{aligned}\lim_{x\to 0}\frac{e^{x^2}-1-x^2}{x^4} &\overset{"\frac{0}{0}"}{=} \lim_{x\to 0}\frac{2x\left(e^{x^2}-1\right)}{4x^3} = \\ &= \lim_{x\to 0}\frac{e^{x^2}-1}{2x^2} \overset{"\frac{0}{0}"}{=} \\ &\underset{t=x^2}{=} \frac{1}{2}\lim_{t\to 0}\frac{e^{t}-1}{t} = \\ &= \frac{1}{2}\ln(e)=\frac{1}{2}\end{aligned}$$

$$\lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{\sin^4(x)} = \lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{x^4}\cdot\frac{x^4}{\sin^4(x)}$$

We know how to compute the limit of the second term, therefore we now focus on the first term.

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{x^4} &\overset{"\frac{0}{0}"}{=} \lim_{x\to 0}\frac{-\frac{1}{1-x}+1+x+x^2}{4x^3} \overset{"\frac{0}{0}"}{=} \\ &= \lim_{x\to 0}\frac{-\frac{1}{(1-x)^2}+1+2x}{12x^2} \overset{"\frac{0}{0}"}{=} \\ &= \lim_{x\to 0}\frac{-\frac{2}{(1-x)^3}+2}{24x} \overset{"\frac{0}{0}"}{=} \\ &= \lim_{x\to 0}\frac{-\frac{6}{(1-x)^3}}{24}=-\frac{1}{4}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{\sin^4(x)} &= \lim_{x\to 0}\frac{\ln(1-x)+x+\frac{1}{2}x^2+\frac{1}{3}x^3}{x^4}\cdot\frac{x^4}{\sin^4(x)} = \\ &= -\frac{1}{4}\cdot 1=-\frac{1}{4}\end{aligned}$$

EXERCISE 2.31. Suppose that in a certain community, there will be $M(r)$ thousand new houses built when the 30-year fixed mortgage rate is $r\%$, where

$$M(r)=\begin{cases}1-e^{r}, & r\le 0,\\ r+r^2, & r>0.\end{cases}$$

Find the interest rate r_{min} such that the number of new houses built is minimized. Is the function differentiable in r_{min}?

Solution. One can see that the function is strictly positive whenever $r \neq 0$. When $r > 0$, the function is an increasing function, since $M'(r) = 2r+1$, and when $r < 0$ the function is a decreasing as $M'(r) = -e^r$. Thus, the minimum should be in $r_{min} = 0$. We can see that $M(0) = 0$, which implies it is minimal. The function is continuous in $r_{min} = 0$ as

$$\begin{aligned} \lim_{r\to 0^+} 1 - e^r &= 1 - 1 = 0; \\ \lim_{r\to 0^-} r + r^2 &= 0. \end{aligned}$$

Using the definition for differentiation we get

$$\lim_{r\to 0} \frac{M(r) - M(0)}{r} = \lim_{r\to 0} \frac{M(r)}{r} =?$$

Taking the two sides separately,

$$\begin{aligned} \lim_{r\to 0^+} \frac{M(r)}{r} &= \lim_{r\to 0^+} \frac{r + r^2}{r} = \lim_{r\to 0^+} 1 + r = 1; \\ \lim_{r\to 0^-} \frac{M(r)}{r} &= \lim_{r\to 0^+} \frac{1 - e^r}{r} \overset{\frac{1}{0^+}}{=} \infty; \end{aligned}$$

therefore, the derivative does not exist.

Exercise 2.32. Determine whether the following function is differentiable at $x_0 = 0$,

$$f(x) = \begin{cases} 1 + \sin(x), & x \geq 0, \\ x - x^2, & x < 0. \end{cases}$$

Solution. One can see that the function is not continuous in $x_0 = 0$ as

$$\begin{aligned} \lim_{x\to 0^+} 1 + \sin(x) &= 1; \\ \lim_{x\to 0^-} x - x^2 &= 0, \end{aligned}$$

and the limit does not exists. Therefore, the function cannot be differentiable.

Exercise 2.33. Determine whether the following function is differentiable at $x_0 = 0$,

$$f(x) = \begin{cases} 1 + \sin(x), & x \geq 0, \\ 1 + x - x^2, & x < 0. \end{cases}$$

Solution. In contrast to the previous exercise, this function is continuous everywhere. Let us compute it derivative through the basic definition. Note that $f(0) = 1 + \sin(0) = 1$, thus

$$\begin{aligned} \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0^+} \frac{1 + \sin(x) - 1}{x - 0} \\ &= \lim_{x\to 0^+} \frac{\sin(x)}{x} = 1; \\ \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0^-} \frac{1 + x - x^2 - 1}{x - 0} \\ &= \lim_{x\to 0^-} \frac{x - x^2}{x} \\ &= \lim_{x\to 0^-} 1 - x = 1, \end{aligned}$$

and the function is differentiable everywhere with a derivative of $f'(0) = 1$.

Exercise 2.34. Consider the function $f(x) = \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2$.

(1) Prove that $f'(x) > 0$ for every $x > 1$.
(2) Assume that $f(x)$is revertible when $x > 1$. compute $x_0 = f^{-1}(0)$.
(3) Compute $\left(f^{-1}\right)'(0)$.

Solution.

(1) A direct computation shows that

$$\begin{aligned} f'(x) &= x\ln(x) + \frac{1}{2}x^2 \cdot \frac{1}{x} - \frac{1}{2}x \\ &= x\ln(x) > 0, \end{aligned}$$

as $x > 1$.

(2) We wish to find x_0 such that $f(x_0) = 0$.

$$\begin{aligned} \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 &= 0 \\ 2x^2\ln(x) - x^2 &= 0 \\ x^2(2\ln(x) - 1) &= 0 \\ x_1 &= 0 \\ 2\ln(x_2) &= 1 \\ x_2 &= e^{1/2}. \end{aligned}$$

Clearly, the function $f(x)$ is not defined when $x = 0$, thus the answer is $x_0 = e^{1/2}$.

(3) Using the formula for the derivative of an inverse function yields

$$\begin{aligned} \left(f^{-1}\right)'(0) &= \frac{1}{f'\left(f^{-1}(0)\right)} \\ &= \frac{1}{f'\left(e^{1/2}\right)} = \\ &= \frac{1}{e^{0.5}\ln\left(e^{0.5}\right)} = \frac{2}{\sqrt{e}}. \end{aligned}$$

2.7. Graphs of functions

In many cases, we can better understand the way a function acts by drawing its graph. Specifically, for every point $x \in \mathbb{R}$, we draw the value of the function $f(x)$ on a two-dimensional space $\mathbb{R}^2$, when the axes are x and y. Note that the graph is only a graphic representation of the function. Figure 2.7.1 present a few graphs of the previously-mentioned functions.

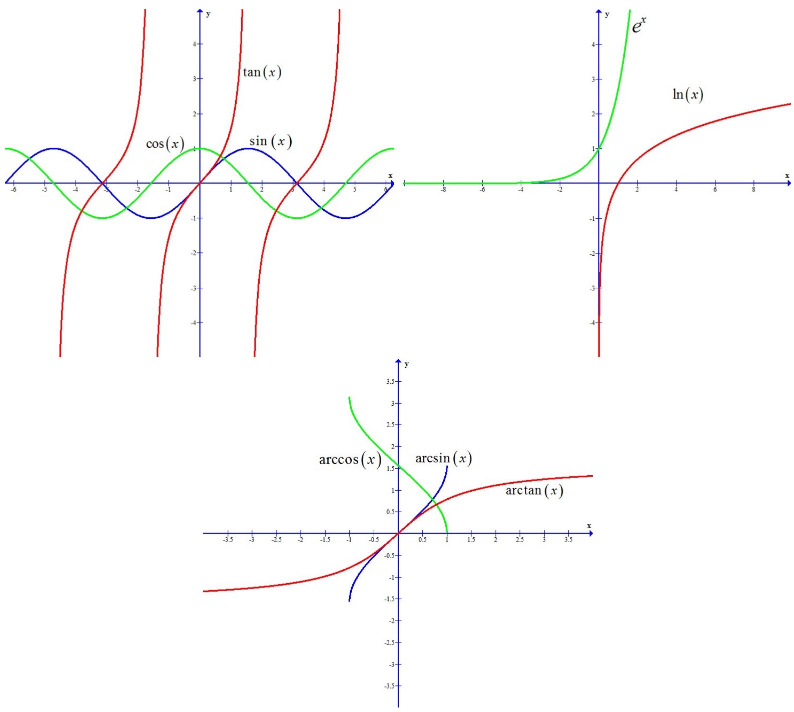

FIGURE 2.7.1. Graphs of commonly-used functions.

2.7.1. Asymptotes.

In order to sketch a graph of a function correctly, we first need to know how to identify asymptotes. There are two types of asymptotes: vertical and non-vertical. A vertical asymptote exists in x_0 if and only if $\lim_{x\to x_0^+} f(x) = \pm\infty$ or $\lim_{x\to x_0^-} f(x) = \pm\infty$. A non-vertical asymptote is of the form $y = ax+b$, and it exists if $\lim_{x\to\infty}(f(x) - (ax+b)) = 0$. Since finding a vertical asymptotes is relatively easy, we focus on the non-vertical ones.

(1) Define $a = \lim_{x\to\infty}\frac{f(x)}{x}$, and $b = \lim_{x\to\infty}(f(x) - ax)$.
(2) If both limits exists and finite then $y = ax+b$ is the non-vertical asymptote.
(3) Otherwise, there is no non-vertical asymptote.
(4) The same computation should be made when $x \to -\infty$.

2.7.2. The stages of sketching the graph of a function.

The final product of all the analysis so far is the graph of a function. Given all the properties we discussed, one should be able to get a good understanding of the graph of a function f. The stages of sketching a graph of a function f are as follows:

(1) Domain D of f.
(2) Intersections with the axis.
(3) Asymptotes.
(4) Extreme points and intervals on which the function is increasing or decreasing.
(5) Convexity and concavity.

EXERCISE 2.35. A business manager determines that t months after production begins on a new product, the number of units produced will be $P(t) = \frac{t}{(t+1)^2}$ million per month. Sketch the graph of P and see what happens to production in the long run (as $t \to \infty$).

Solution.

(1) The function is defined for all real $t \neq -1$.
(2) The only intersection point is when $t = 0$, and we get the point $(0,0)$.
(3) The function is continuous everywhere except $t = -1$, thus we need to check for an asymptote in $t = -1$.

$$\begin{aligned}\lim_{t\to -1^+}\frac{t}{(t+1)^2} &= -\infty;\\ \lim_{t\to -1^-}\frac{t}{(t+1)^2} &= -\infty.\end{aligned}$$

For non-vertical asymptotes, we compute

$$\begin{aligned}a_+ &= \lim_{t\to\infty}\frac{P(t)}{t} = \lim_{t\to\infty}\frac{1}{(t+1)^2} = 0,\\ a_- &= \lim_{t\to-\infty}\frac{P(t)}{t} = \lim_{t\to-\infty}\frac{1}{(t+1)^2} = 0.\end{aligned}$$

And

$$b_\pm = \lim_{t\to\pm\infty} P(t) - 0 = \lim_{t\to\pm\infty}\frac{t}{(t+1)^2} \overset{L}{=} \lim_{t\to\pm\infty}\frac{1}{2(t+1)} = 0.$$

Thus, $t = -1$ and $y = 0$ are the two asymptotes.

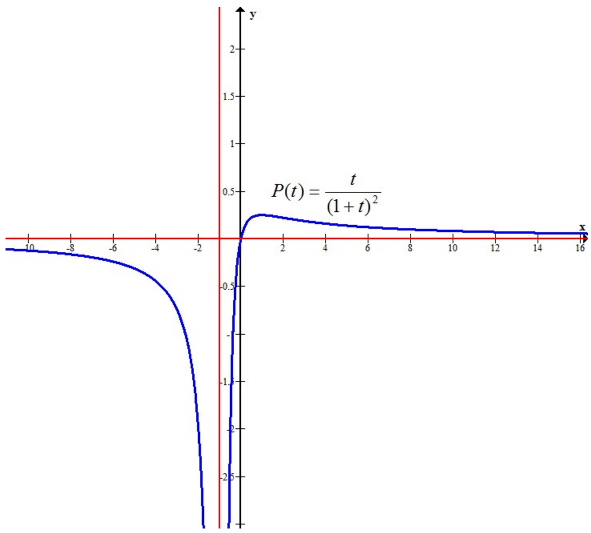

FIGURE 2.7.2. The graph of the function in Exercise 2.35

(4) We need to find the first and second derivatives of the function.

$$\begin{aligned} P'(t) &= \frac{(t+1)^2 - 2t(t+1)}{(1+t)^4} \\ &= \frac{(t+1) - 2t}{(1+t)^3} \\ &= \frac{1-t}{(1+t)^3}; \\ P''(t) &= \frac{-(1+t)^3 - 3(1-t)(1+t)^2}{(1+t)^6} \\ &= \frac{-(1+t) - 3(1-t)}{(1+t)^4} \\ &= \frac{2t-4}{(1+t)^4}. \end{aligned}$$

The critical point are given by $P'(t) = 0$, hence

$$\frac{1-t}{(1+t)^3} = 0 \quad \Rightarrow t = 1.$$

The value of the function at $t = 1$ is $P(1) = \frac{1}{4}$. So, we have an extreme point $\left(1, \frac{1}{4}\right)$. Note that the function is increasing for every

$$1 - t \geq 0 \text{ and } t + 1 > 0,$$

hence when $t \in (-1, 1)$. Otherwise, Either the denominator is negative or the numerator is positive, or the other way around, such that, overall, the derivative is negative and the function is decreasing.

(5) The second derivative is given by $P''(t) = \frac{2t-4}{(1+t)^4}$. This means that the function is concave when $2t - 4 < 0$, or equivalently $t < 2$. That is, the function is convex when $t > 2$, and $\left(1, \frac{1}{4}\right)$ is a maximum point. There is an inflection pint in $\left(2, \frac{2}{9}\right)$.

EXERCISE 2.36. The total cost of producing x units of a particular commodity is $C(x) = \left|\frac{e^{-x} - e}{1-x}\right|$ thousand dollars. Sketch the graph of $C(x)$. (No need to analyze the second derivative).

Solution.

(1) The function is defined for all real $x \neq 1$.

(2) One intersection point is when $e^{-x} = e$, meaning $x = -1$, and we get the point $(-1, 0)$. Another intersection point is when $x = 0$, and we get $(0, e-1)$. Note that the function is always positive, so we can represent it as follows $C(x) = \left|\frac{e-e^{-x}}{x-1}\right|$

$$C(x) = \begin{cases} \frac{e-e^{-x}}{1-x}, & -1 \leq x \leq 1, \\ \frac{e^{-x}-e}{1-x}, & \text{otherwise.} \end{cases}$$

(3) The function is continuous everywhere except $x = 1$, thus we need to check for an asymptote in $x = 1$.

$$\lim_{x \to 1^{\pm}} \left|\frac{e^{-x} - e}{1 - x}\right| \overset{\text{"}\left|\frac{\frac{1}{e}-e}{0}\right|\text{"}}{=} \infty.$$

For non-vertical asymptotes, we compute

$$\begin{aligned} a_{+} &= \lim_{x \to \infty} \frac{C(x)}{x} = \lim_{x \to \infty} \frac{\left|\frac{e^{-x}-e}{1-x}\right|}{x} \overset{\text{"}\frac{0}{\infty}\text{"}}{=} 0, \\ a_{-} &= \lim_{x \to -\infty} \frac{C(x)}{x} = \lim_{x \to -\infty} \frac{\left|\frac{e^{-x}-e}{1-x}\right|}{x} = \lim_{x \to -\infty} \frac{\frac{e^{-x}-e}{1-x}}{x} = \lim_{x \to -\infty} \frac{e^{-x} - e}{x - x^2} \\ &= \lim_{t \to \infty} \frac{e^t - e}{-t - t^2} \overset{L}{=} \lim_{t \to \infty} \frac{e^t}{-1 - 2t} \overset{L}{=} \lim_{t \to \infty} \frac{e^t}{-2} = \infty. \end{aligned}$$

And

$$b_{+} = \lim_{x \to \infty} C(x) - 0 = \lim_{x \to \infty} \frac{e^{-x} - e}{1 - x} \overset{L}{=} \lim_{x \to \infty} \frac{-e^{-x}}{-1} = 0.$$

Thus, $x = 1$ and $y = 0$ (when $x \to \infty$) are the only two asymptotes.

(4) We need to find the first derivative of the function. We cannot use simple differentiation in $x = \pm 1$, but we can differentiate separately each interval.

$$\begin{aligned} C'(x) &= \begin{cases} \frac{e^{-x}(1-x)+e-e^{-x}}{(1-x)^2}, & -1 < x < 1, \\ \frac{-e^{-x}(1-x)+e^{-x}-e}{(1-x)^2}, & x > 1 \text{ or } x < -1, \end{cases} \\ &= \begin{cases} \frac{e-xe^{-x}}{(1-x)^2}, & -1 < x < 1, \\ \frac{xe^{-x}-e}{(1-x)^2}, & x > 1 \text{ or } x < -1. \end{cases} \end{aligned}$$

The critical point are given by $C'(x) = 0$, hence

$$xe^{-x} - e = 0 \Leftrightarrow \frac{x}{e^x} - e = 0.$$

When $x < 0$, the right-hand side is negative as well. When $x > 1$, $\frac{x}{e^x} < 1 < e$, and the RHS is still negative. When $x \in (0, 1)$, then $x - e^{x+1} < 0$ and the RHS is still negative. Thus, the function $xe^{-x} - e$ is always negative. This implies that

$$C'(x) = \begin{cases} > 0 & -1 < x < 1, \\ < 0, & x > 1 \text{ or } x < -1, \end{cases}$$

and the function is increasing if and only if $x \in (-1, 1)$. Note that the function is not differentiable in $x = -1$, as the derivative from the right is negative whereas the derivative from the left is positive.

EXERCISE 2.37. Sketch the graph of the function $f(x) = \frac{1}{4}x^4 - \frac{3}{2}x^3 + 3$.

Solution.

(1) Since this a polynomial the domain is every $x \in \mathbb{R}$.

(2) We need to solve the equation

$$\frac{1}{4}x^4 - \frac{3}{2}x^3 + 3 = 0.$$

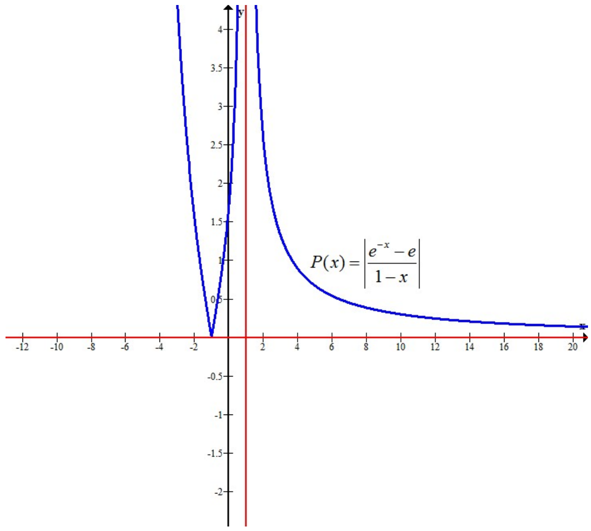

FIGURE 2.7.3. The graph of the function in Exercise 2.36

Since this is a problematic equation, we will get to that later. However, we do know that $f(0) = 3$, which means that there is an intersection in , $(0, 3)$.

(3) Taking the FOC

$$\begin{aligned} f'(x) &= x^3 - \frac{9}{2}x^2 = 0 \\ x_1 &= 0, \\ x_2 &= \frac{9}{2}. \end{aligned}$$

We can plug in x_1 and x_2 in f and get the exact coordinates of the extreme points. When $x < \frac{9}{2}$, the derivative is negative and the function is decreasing. When $x > \frac{9}{2}$, the function is increasing. The second derivative is

$$f''(x) = 3x^2 - 9x.$$

Thus, $x = 0$ is neither a maximum nor a minimum, and $x = \frac{9}{2}$ is a local minimum.

(4) When $x < 0$, $f''(x) > 0$ which means that the function is convex and the same holds for $x > 3$. Otherwise, the function is concave.

(5) As this is a polynomial, there are no vertical asymptotes. We can see that there are no non-vertical asymptotes as well, because $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$.

EXERCISE 2.38. Sketch the graph of the function $f(x) = \frac{x^2+1}{2x}$.

Solution.

(1) The function is defined for every real-valued $x \neq 0$.

(2) We need to solve the equation

$$\frac{x^2+1}{2x} = 0.$$

We can see that $x^2 + 1 > 0$ for every x, and this means that the equation does not have a solution.

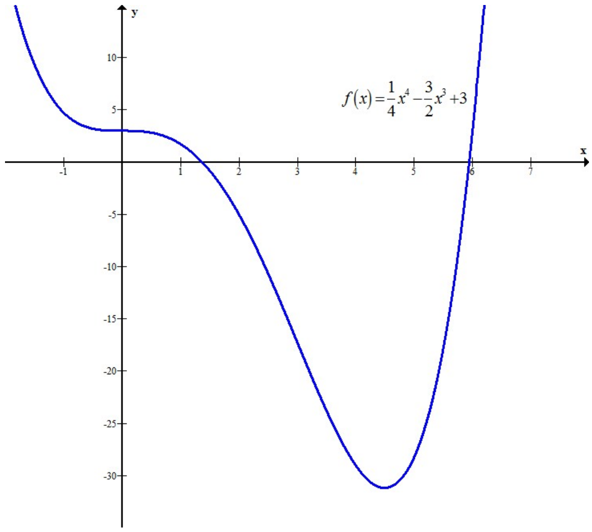

FIGURE 2.7.4. The graph of the function in Exercise 2.37

(3) Taking the FOC

$$\begin{aligned} f'(x) &= \frac{4x^2-2x^2-2}{4x^2} \\ &= \frac{2x^2-x^2-1}{2x^2} \\ &= \frac{x^2-1}{2x^2} \\ &= \frac{1}{2}-\frac{1}{2x^2}=0. \\ x_1 &= 1, \\ x_2 &= -1. \end{aligned}$$

We can plug in x_1 and x_2 in f and get the exact coordinates of the extreme points. When $x \in (-1,1)$, the derivative is negative and the function is decreasing. When $x > 1$ or $x < -1$, the function is increasing. The second derivative is

$$f''(x) = x^{-3}.$$

Thus, $x = 1$ is a local minimum, and $x = -1$ is a local maximum.

(4) When $x < 0$, $f''(x) < 0$ which means that the function is concave. Otherwise, the function is convex.

(5) We know that we have a vertical asymptote in $x = 0$ because

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{2}+\frac{1}{2x} = \infty.$$

In addition, there is a non-vertical asymptote as

$$\begin{aligned} a &= \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{1}{2}+\frac{1}{2x^2} = \frac{1}{2} \\ ,b &= \lim_{x\to\infty} (f(x)-ax) = \lim_{x\to\infty} \left(\frac{x}{2}+\frac{1}{2x}-\frac{1}{2}x\right) = 0 \end{aligned}$$

and

$$\begin{aligned} a &= \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \frac{1}{2}+\frac{1}{2x^2} = \frac{1}{2} \\ .b &= \lim_{x\to-\infty} (f(x)-ax) = \lim_{x\to-\infty} \left(\frac{x}{2}+\frac{1}{2x}-\frac{1}{2}x\right) = 0. \end{aligned}$$

This implies that $y = \frac{1}{2}x$ is an asymptote.

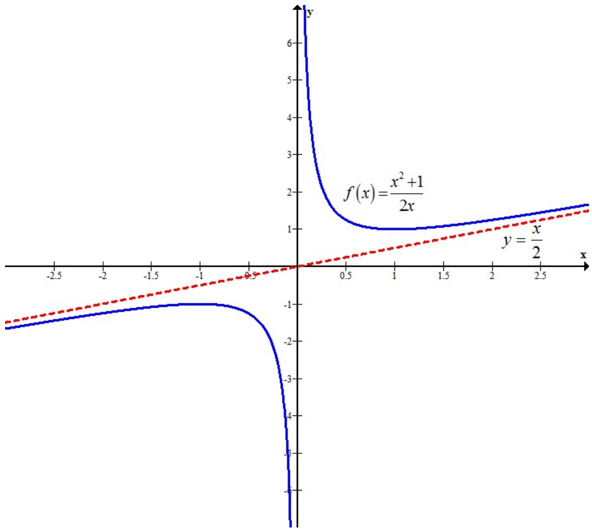

FIGURE 2.7.5. The graph of the function in Exercise 2.38

EXERCISE 2.39. Sketch the graph of the function

$$.f(x) = \sqrt{x^2+1} - \frac{1}{\sqrt{2}}x.$$

Solution.
Note that $x^2+1>0$ for every x hence the function is defined for every $x \in \mathbb{R}$. The function has a single intersection with the y axis and it is $(0,1)$, There are no other intersections with the axis. Computing the derivatives of the function yields ,

$$f'(x) = \left[\sqrt{x^2+1} - \frac{1}{\sqrt{2}}x\right]' = \frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}$$

$$.f''(x) = \left[\frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}}\right]' = \frac{1}{\sqrt{x^2+1}} - \frac{x^2}{(x^2+1)^{\frac{3}{2}}} = \frac{1}{(x^2+1)^{\frac{3}{2}}}$$

Let us compare the first derivative to0 in order to find extreme points,

$$\begin{aligned} f'(x) = \frac{x}{\sqrt{x^2+1}} - \frac{1}{\sqrt{2}} &= 0 \\ \sqrt{2}x &= \sqrt{x^2+1} \\ 2x^2 &= x^2+1 \\ x^2 &= 1 \\ .x_{1,2} &= \pm\ 1 \end{aligned}$$

We got 2 critical points where the derivative is zero. After plugging in the values we can see that the derivative is only 0 when $x=1$. When $x<1$ the derivative is negative and the function is decreasing. Otherwise the function is increasing. Let's sum this up in a table.

	decreasing	local minimum	increasing
$f'(x)$	-	0	+
x	$x<1$	$x=1$	$x>1$

We now try to find whether the function is convex or concave using the second derivative. We can see that - $f''(x)>0$ always and the function is convex.

We do not have any vertical asymptotes since the function is continuous. Computing the limits for non-vertical asymptotes yields

$$\begin{aligned}\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\sqrt{2x^2+2}-x}{\sqrt{2}} &= \lim_{x\to\infty}\frac{x^2+2}{\sqrt{2}\left(\sqrt{2x^2+2}+x\right)} = \\ &= \lim_{x\to\infty}\frac{1+\frac{2}{x^2}}{\sqrt{2}\left(\frac{\sqrt{2x^2+2}}{x^2}+\frac{1}{x}\right)} = \infty \\ \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\sqrt{2x^2+2}-x}{\sqrt{2}} &= \infty\end{aligned}$$

$$\begin{aligned} a &= \lim_{x\to\infty}\frac{\frac{\sqrt{2x^2+2}-x}{\sqrt{2}}}{x} = \lim_{x\to\infty}\frac{\sqrt{2+\frac{2}{x^2}}-1}{\sqrt{2}} = 1-\frac{1}{\sqrt{2}} \\ ,b &= \lim_{x\to\infty}\left(f(x)-ax\right) = \lim_{x\to\infty}\left(\frac{\sqrt{2x^2+2}-x}{\sqrt{2}}-\left(1-\frac{1}{\sqrt{2}}\right)x\right) = \\ &= \lim_{x\to\infty}\left(\frac{\sqrt{2x^2+2}-\sqrt{2}x}{\sqrt{2}}\right) = \lim_{x\to\infty}\left(\frac{2x^2+2-2x^2}{\sqrt{2}}\cdot\frac{1}{\sqrt{2x^2+2}+\sqrt{2}x}\right) = \\ &= \lim_{x\to\infty}\left(\frac{2}{\sqrt{2}\left(\sqrt{2x^2+2}+\sqrt{2}x\right)}\right) = 0\end{aligned}$$

We found that $y=\left(1-\frac{1}{\sqrt{2}}\right)x$ is a non-vertical asymptote of the function. Computing the same limit when $x\to-\infty$ gives out the answer

$$\begin{aligned} a &= \lim_{x\to-\infty}\frac{\frac{\sqrt{2x^2+2}-x}{\sqrt{2}}}{x} = \lim_{x\to-\infty}\frac{-\sqrt{2+\frac{2}{x^2}}-1}{\sqrt{2}} = -1-\frac{1}{\sqrt{2}} \\ ,b &= \lim_{x\to-\infty}\left(f(x)-ax\right) = \lim_{x\to-\infty}\left(\frac{\sqrt{2x^2+2}-x}{\sqrt{2}}-\left(-1-\frac{1}{\sqrt{2}}\right)x\right) = \\ &= \lim_{x\to-\infty}\left(\frac{\sqrt{2x^2+2}+\sqrt{2}x}{\sqrt{2}}\right) = \lim_{x\to-\infty}\left(\frac{2x^2+2-2x^2}{\sqrt{2}}\cdot\frac{1}{\sqrt{2x^2+2}-\sqrt{2}x}\right) = \\ &= \lim_{x\to-\infty}\left(\frac{2}{\sqrt{2}\left(\sqrt{2x^2+2}+\sqrt{2}x\right)}\right) = 0\end{aligned}$$

and $y=\left(-1-\frac{1}{\sqrt{2}}\right)x$ is another non-vertical asymptote. We can now sketch the graph of the function.

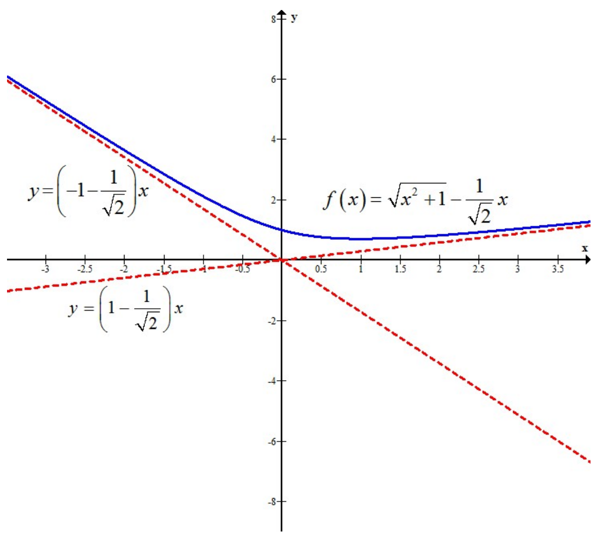

FIGURE 2.7.6. The graph of the function from Exercise 2.39

2.8. Integrals

There are two types of integrals: indefinite integral and definite integral. Although they share a similar name, there definitions are very different. We start with the simpler notion of indefinite integral.

2.8.1. The indefinite integral.

The indefinite integral is basically the inverse of differentiation. That is, if $f(x) = F'(x)$, then $\int f(x)\,dx$ =F$(x)+c$, when c is some constant number. $F(x)$ is sometimes called the anti-derivative of f. [2] There are a few helpful rules when considering indefinite integration. Assume that $F(x) = \int f(x)\,dx$ and $G(x) = \int g(x)\,dx$, then:

- Linearity implies that $\int (f(x) \pm g(x))\,dx = F(x) \pm G(x)$, and $\int cf(x)\,dx = c\int f(x)\,dx$.
- Integration by parts states that $\int f(x) G(x)\,dx = F(x) G(x) - \int F(x) g(x)\,dx$.
- Exchanging the integrating variable x to $h(y)$ such that $\int f(x)\,dx = \int f(h(y)) h'(y)\,dy$. Assume that $x = h(y)$, Then, taking the derivative with respect to y yields $\frac{dx}{dy} = h'(y)$. Thus, $dx = h'(y)\,dy$, and we get the previously stated formula. (Don't forget to return to the original variable x after you finish the computation involving y!)
- Integration and differentiation cancel out each other: $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$, and $\int \frac{d}{dx}[F(x)]\,dx = F(x)$.

REMARK 2.7. *Using Substitution of variables.* In some cases, it is better the change the variables before integrating to simplify the computation. Assume you wish compute $\int f(x)\,dx$. For simplicity, you can use a different variable $x = h(y)$ such that $f(x)\,dx = f(h(y))\,u'(y)\,dy$. Note that we used the three following steps: (i) Choose a substitution $x = h(y)$ that transforms the term $f(x)dx$ into something simpler; (ii) express the entire integral in terms of y and dy. This means that all terms involving x and dx must be transformed to terms involving y and dy; and, (iii) compute the new integral and then return to the original variable x. It is important to return to the original variable, since we are looking for a function whose derivative w.r.t. x, is f, and not with respect to a different variable.

EXERCISE 2.40. Compute the following integrals:

(1) $\int \frac{x^2+2x+1}{x}dx$.
(2) $\int xe^{-x^2}dx$.
(3) $\int xe^{x}dx$.
(4) $\int \ln(x+1)\,dx$.
(5) $\int \frac{1}{a^2+x^2}dx$.
(6) $\int \tan(x)\,dx$.
(7) $\int \sqrt{1-x^2}dx$.
(8) $\int \frac{1}{x^2-4x+8}dx$.
(9) $\int \frac{(\ln(x))^2}{x}dx$.
(10) $\int \frac{1}{1+e^{-x}}dx$.

Solution. We compute these integrals using the previously-mentioned rules and theorems.

(1)

$$\begin{aligned}\int \frac{x^2+2x+1}{x}dx &= \int x+2+\frac{1}{x}dx = \\ &= \frac{x^2}{2}+2x+\ln(x)+C.\end{aligned}$$

[2] Every indefinite integral is determined up to a constant, thus we need to add c after integrating a function.

(2)

$$\begin{aligned}\int xe^{-x^2}dx &\underset{y=x^2,\ dy=2xdx}{=} \int \frac{1}{2}e^{-y}dy = \\ &= -\frac{1}{2}e^{-y}+C= \\ &= -\frac{1}{2}e^{-x^2}+C.\end{aligned}$$

(3)

$$\begin{aligned}\int xe^{x}dx &= xe^{x}-\int e^{x}dx= \\ &= xe^{x}-e^{x}+C.\end{aligned}$$

(4)

$$\begin{aligned}\int \ln(x+1)\,dx &= \int 1\cdot\ln(x+1)\,dx= \\ &= x\ln(x+1)-\int\frac{x}{x+1}dx= \\ &= x\ln(x+1)-\int 1-\frac{1}{x+1}dx= \\ &= x\ln(x+1)-x+\ln(x+1)+C.\end{aligned}$$

(5)

$$\begin{aligned}\int\frac{1}{a^2+x^2}dx &= \int\frac{1}{a^2}\cdot\frac{1}{1+\left(\frac{x}{a}\right)^2}dx= \\ &\underset{x=ay,\ dx=ady}{=} \frac{1}{a^2}\int\frac{1}{1+\left(\frac{ay}{a}\right)^2}ady= \\ &= \frac{a}{a^2}\int\frac{1}{1+y^2}dy= \\ &= \frac{1}{a}\arctan(y)+C= \\ &= \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C.\end{aligned}$$

(6)

$$\begin{aligned}\int\tan(x)\,dx &= \int\frac{\sin(x)}{\cos(x)}dx= \\ &\underset{\cos(x)=y,\ dy=-\sin(x)dx}{=} \int\frac{\sin(x)}{y}\cdot\left(-\frac{dy}{\sin(x)}\right)= \\ &= -\int\frac{dy}{y}= \\ &= -\ln|y|+C= \\ &= -\ln|\cos(x)|+C.\end{aligned}$$

(7)

$$\begin{aligned}
\int \sqrt{1-x^2}dx & \underset{\substack{x=\sin(t)\\ dx=\cos(t)\,dt}}{=} \int \sqrt{1-\sin(t)^2}\cos(t)\,dt = \\
& = \int \cos^2(t)\,dt = \\
& = \frac{1}{2}\int (\cos(2t)+1)\,dt = \\
& = \frac{1}{2}t + \frac{1}{2}\int \cos(2t)\,dt = \\
& = \frac{1}{2}t + \frac{1}{4}\sin(2t) + C = \\
& = \frac{1}{2}\arcsin(x) + \frac{1}{4}\cdot 2\sin(t)\cos(t) + C = \\
& = \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C.
\end{aligned}$$

(8)

$$\begin{aligned}
\int \frac{1}{x^2-4x+8}dx & = \int \frac{1}{x^2-4x+4+4}dx = \\
& = \int \frac{1}{(x-2)^2+4}dx = \\
& = \frac{1}{4}\int \frac{1}{\left(\frac{x-2}{2}\right)^2+1}dx = \\
& \underset{y=\frac{x-2}{2},\ dx=2dy}{=} \frac{1}{2}\int \frac{1}{y^2+1}dy = \\
& = \frac{1}{2}\arctan(y) + C = \\
& = \frac{1}{2}\arctan\left(\frac{x-2}{2}\right) + C.
\end{aligned}$$

(9)

$$\begin{aligned}
\int \frac{(\ln(x))^2}{x}dx & \underset{u=\ln(x),\ \frac{du}{dx}=\frac{1}{x},\ \frac{1}{x}dx=du}{=} \int u^2du \\
& = \frac{1}{3}u^3 + C \\
& = \frac{1}{3}(\ln(x))^3 + C.
\end{aligned}$$

(10)

$$\begin{aligned}
\int \frac{1}{1+e^{-x}}dx & = \int \frac{e^x}{e^x+1}dx \\
& \overset{u=e^x,\ e^xdx=du}{=} \int \frac{1}{u+1}du \\
& = \ln|u+1| + C \\
& = \ln|e^x+1| + C.
\end{aligned}$$

2.8.2. The definite integral. The definite integral is a way of calculating the area between a curve and the x axis. The formal method is based on dividing an interval to many small intervals and computing the area between each interval and the function. The idea behind this formula is simple, yet applying it might be complicated.

The basic areas we know how to compute are rectangles. We can always take the product of the height with the base to get the area. Consider a function f and assume we wish to evaluate the area below

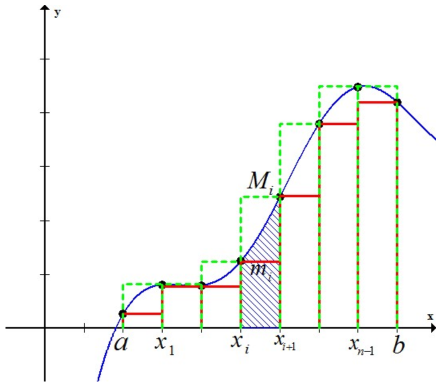

FIGURE 2.8.1. The approximation of the area under a curve using rectangles.

this function bounded by the interval $[a,b]$. That is, when $x \in [a,b]$. Since the function might be a complicated one, we can divide the interval into small sub intervals and compute the area of a rectangle bounded by the lowest value of the function in that sub interval and the height of the highest value. Denote the number of interval by n, where each is of the uniform length $h = \frac{b-a}{n}$, and denote the highest and lowest values by M_i and m_i respectively, considering the sub interval $i = 1, \ldots, n$.
We can now produce two sums

$$,\underline{S_n} = \sum_{i=0}^{n-1} m_i \cdot h = \sum_{i=0}^{n-1} m_i \cdot \left(\frac{b-a}{n}\right) \quad , \quad .\overline{S_n} = \sum_{i=0}^{n-1} M_i \cdot h = \sum_{i=0}^{n-1} M_i \cdot \left(\frac{b-a}{n}\right)$$

such that the true value of the area S is bounded by $\underline{S_n} \le S \le \overline{S_n}$ for every natural n. Thus, in case $\lim_{n\to\infty} \underline{S_n} = \lim_{n\to\infty} \overline{S_n}$, we say that the definite integral exists and the required area equals the limit. Nevertheless, this method tends to be very complicated and therefore, we commonly use the Newton-Leibniz formula which combines between indefinite integration and the definite one. This formula is also called the fundamental theorem of calculus:

THEOREM 2.8. ***(The Newton-Leibniz formula)*** *Let f be a continuous integrable function over the interval $[a,b]$ and assume that F is an anti-derivative of the function f, then:*

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a).$$

The definite integral also has a few assisting rules. Let $f(x), g(x)$ be two integrable function on the interval $[a,b]$.

(1) Linearity - $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.
(2) For every $k \in \mathbb{R}$, it holds that $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
(3) For every $c \in [a,b]$ it follows that $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.
(4) If $f(x) \le g(x)$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.
(5) if $f(x) \le 0$ in $[a,b]$,then $\int_a^b f(x)\,dx = -S$.
(6) $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.
(7) $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

EXERCISE 2.41. A manufacturer has found that marginal cost is $3x^2 - 60x + 400$ dollars per unit when x units have been produced. The total cost of producing the first two units is \$900. What is the total cost of producing the first 5 units?

Solution. Recall that the marginal cost is the derivative of the total cost function $C(x)$. Thus,

$$\frac{dC}{dx} = 3x^2 - 60x + 400,$$

and

$$C(x) = \int 3x^2 - 60x + 400 = x^3 - 30x^2 + 400x + K.$$

Trying to estimate K, we get $C(2) = 900 = 2^3 - 30 \cdot 2^2 + 400 \cdot 2 + K$, therefore $K = 212$. Now, we can use the formula to get

$$C(5) = 5^3 - 30 \cdot 5^2 + 400 \cdot 5 + 212 = \$1587.$$

Similarly to the previous mean value theorem we discussed, there is a mean value theorem for integrals.

THEOREM 2.9. **(The Mean Value Theorem for integrals)** *Let f be a continuous function on $[a, b]$. Then there exists a point $c \in (a, b)$ such that*

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

EXERCISE 2.42. Suppose that t years from now, one investment will be generating profit at the rate of $P_1'(t) = 50 + t^2$ hundred dollars per year, while a second investment will be generating profit at the rate of $P_2'(t) = 200 + 5t$ hundred dollars per year.

(1) For how many years does the rate of profitability of the second investment exceed that of the first?
(2) Compute the net excess profit for the time period determined in the previous question.

Solution.

(1) The rate of profitability of the second investment exceeds that of the first until

$$\begin{aligned} P_1'(t) &= P_2'(t) \\ 50 + t^2 &= 200 + 5t \\ t^2 - 5t - 150 &= 0 \\ (t-15)(t+10) &= 0 \\ t_1 = -10 \quad &, \quad t_2 = 15. \end{aligned}$$

So the answer is $t = 15$ years.

(2) We need to compute the net excess profit of one investment over the other.

$$\begin{aligned} \int_0^{15} P_2'(t)\,dt - \int_0^{15} P_1'(t)\,dt &= \int_0^{15} [P_2'(t) - P_1'(t)]\,dt \\ &= \int_0^{15} \left[150 + 5t - t^2\right] dt \\ &= \left[150t + \frac{5}{2}t^2 - \frac{1}{3}t^3\right]_0^{15} \\ &= 1687.5. \end{aligned}$$

Thus, the net excess profit is $\$168,750$.

EXERCISE 2.43. Compute the following integrals:

(1) $_{-1}\int^1 x \arctan(x)\,dx$.
(2) $_{-\pi/6}\int^{\pi/6} \cos(x) \ln(\sin(x) + 1)\,dx$.

Solution. Using the Newton-Leibniz formula, Theorem 2.8, we solve every integral separately.

$$
\begin{aligned}
1.\ \int_{-1}^{1} x\arctan(x)\,dx &= \left[\frac{x^2}{2}\arctan(x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx = \\
&= \frac{1}{2}\arctan(1) - \frac{1}{2}\arctan(-1) - \frac{1}{2}\int_{-1}^{1}\frac{x^2+1-1}{1+x^2}dx = \\
&= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{2}\cdot\frac{-\pi}{4} - \frac{1}{2}\int_{-1}^{1}\frac{x^2+1}{1+x^2}dx + \frac{1}{2}\int_{-1}^{1}\frac{1}{1+x^2}dx = \\
&= \frac{\pi}{4} - \frac{1}{2}\int_{-1}^{1}1\cdot dx + \frac{1}{2}\arctan(x)\,|_{-1}^{1} = \\
&= \frac{\pi}{4} - \frac{1}{2}x|_{-1}^{1} + \frac{1}{2}(\arctan(1) - \arctan(-1)) = \\
&= \frac{\pi}{4} - \frac{1}{2}(1-(-1)) + \frac{1}{2}\left(\frac{\pi}{4} - \frac{-\pi}{4}\right) = \\
&= \frac{\pi}{2} - 1.
\end{aligned}
$$

$$
\begin{aligned}
2.\ \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos(x)\ln(\sin(x)+1)\,dx &\underset{\substack{t=\sin(x)+1 \\ dt=\cos(x)\,dx \\ x=\frac{\pi}{6}\Rightarrow t=\frac{3}{2} \\ x=-\frac{\pi}{6}\Rightarrow t=\frac{1}{2}}}{=} \int_{\frac{1}{2}}^{\frac{3}{2}}\ln(t)\,dt = \\
&= \int_{\frac{1}{2}}^{\frac{3}{2}} 1\cdot\ln(t)\,dt = \\
&= [t\cdot\ln(|t|)]_{0.5}^{1.5} - \int_{\frac{1}{2}}^{\frac{3}{2}} t\cdot\frac{1}{t}dt = \\
&= 1.5\ln(1.5) - 0.5\ln(0.5) - t|_{0.5}^{1.5} = \\
&= 1.5\ln(1.5) - 0.5\ln(0.5) - 1.5 + 0.5 = \\
&= \frac{3}{2}\ln\left(\frac{3}{2}\right) - \frac{1}{2}\ln\left(\frac{1}{2}\right) - 1
\end{aligned}
$$

EXERCISE 2.44. Prove that

$$1 \leq_0 \int^1 e^{\sin\left(\ln\left(x^2+1\right)\right)}dx \leq 3.$$

PROOF. We will use the Main Value Theorem, Theorem 2.9, in this proof. Note that for every $0 \leq x \leq 1$,

$$, 1 \leq e^{\sin\left(\ln\left(x^2+1\right)\right)} \leq e < 3$$

as x^2+1 is a monotone increasing function, $0 \leq \ln\left(x^2+1\right) \leq 0.7$ is also monotonically increasing when $0 \leq x \leq 1$ and so $0 \leq \sin\left(\ln\left(x^2+1\right)\right) \leq 0.65$ is monotonically increasing when $0 \leq x \leq 1$. From the mean value theorem for integrals there exists $0 \leq c \leq 1$ such that

$$, e^{\sin\left(\ln\left(c^2+1\right)\right)} = \frac{1}{1-0}\int_0^1 e^{\sin\left(\ln\left(x^2+1\right)\right)}dx$$

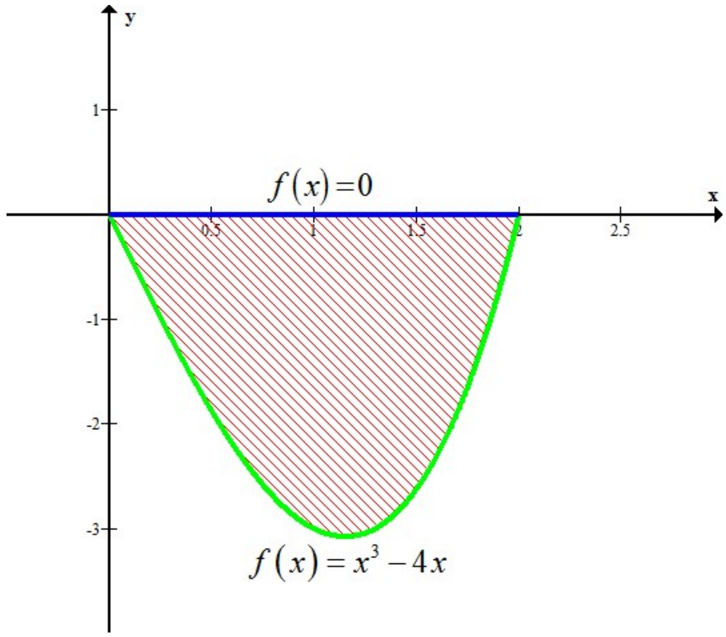

FIGURE 2.8.2. The area between $y = x^3 - 4x$ and the x axis is highlighted in red.

Thus$1 \le e^{\sin\left(\ln\left(c^2+1\right)\right)} \le e$ and the result follows. □

EXERCISE 2.45. Compute the area between the curves $y = x^3 - 4x$, $y = 0$, $x = 0$, and $x = 2$.

Solution. First we sketch the graphs of the functions and the points where they intersect, see Figure 2.8.2.
It is easy to see that for every $x \in [0, 2]$ the function $f(x) = x^3 - 4x$ is negative and so

$$.S = \int\limits_0^2 \left(0 - x^3 + 4x\right) dx = \left[-\frac{x^4}{4} + \frac{4x^2}{2}\right]_0^2 = 4$$

EXERCISE 2.46. Compute the following improper integrals:

(1) ${}_2\int^{\infty} \frac{1}{x\ln(x)} dx.$
(2) ${}_2\int^{\infty} \frac{1}{x\ln^2(x)} dx.$
(3) $\int_1^{\infty} \frac{1}{(x-1)^{\frac{1}{3}}} dx.$
(4) $\int_0^1 \frac{e^x}{(e^x-1)^{\frac{1}{2}}} dx.$

Solution. We will solve every integral separately and then use limits to compute the improper integrals.

$$1. \quad \int\limits_2^{\infty} \frac{1}{x\ln(x)} dx = \lim_{b\to\infty} \int\limits_2^{b} \frac{1}{x\ln(x)} dx$$

$$\int\limits_2^{b} \frac{1}{x\ln(x)} dx \underset{t=\ln(x),dt=\frac{1}{x}dx}{=} \int\limits_{\ln(2)}^{\ln(b)} \frac{1}{t} dt = $$

$$= \ln(\ln(b)) - \ln(\ln(2)).$$

And so we get that

$$\int\limits_2^{\infty} \frac{1}{x\ln(x)} dx = \lim_{b\to\infty} \int\limits_2^{b} \frac{1}{x\ln(x)} dx$$

$$= \lim_{b\to\infty} \left[\ln(\ln(b)) - \ln(\ln(2))\right] = \infty,$$

which means that the integral diverges.

$$\begin{aligned}
2.\ \int_{2}^{\infty}\frac{1}{x\ln^{2}(x)}dx &= \lim_{b\to\infty}\int_{2}^{b}\frac{1}{x\ln^{2}(x)}dx\\
\int_{2}^{b}\frac{1}{x\ln(x)}dx &\underset{t=\ln(x),dt=\frac{1}{x}dx}{=} \int_{\ln(2)}^{\ln(b)}\frac{1}{t^{2}}dt=\\
&= \left[-\frac{1}{t}\right]_{\ln(2)}^{\ln(b)}=\\
&= \left[-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}\right],
\end{aligned}$$

so, we see that

$$\begin{aligned}
\int_{2}^{\infty}\frac{1}{x\ln^{2}(x)}dx &= \lim_{b\to\infty}\int_{2}^{b}\frac{1}{x\ln^{2}(x)}dx\\
&= \lim_{b\to\infty}\left[-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}\right]=\\
&= \frac{1}{\ln(2)},
\end{aligned}$$

and the integral converges.

Concerning the third exercise, we first compute the following integral

$$\begin{aligned}
3.\ \int_{1}^{2}\frac{1}{(x-1)^{\frac{1}{3}}}dx &= \lim_{a\to1^{+}}\int_{a}^{2}\frac{1}{(x-1)^{\frac{1}{3}}}dx=\\
&= \lim_{a\to1^{+}}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_{a}^{2}=\\
&= \frac{3}{2}\lim_{a\to1^{+}}\left[(2-1)^{\frac{2}{3}}-(a-1)^{\frac{2}{3}}\right]=\\
&= \frac{3}{2}\lim_{a\to1^{+}}\left[1-(a-1)^{\frac{2}{3}}\right]=\frac{3}{2}.
\end{aligned}$$

Now, we get

$$\begin{aligned}
\int_{1}^{\infty}\frac{1}{(x-1)^{\frac{1}{3}}}dx &= \int_{1}^{2}\frac{1}{(x-1)^{\frac{1}{3}}}dx+\int_{2}^{\infty}\frac{1}{(x-1)^{\frac{1}{3}}}dx=\\
&= \frac{3}{2}+\lim_{b\to\infty}\int_{2}^{b}\frac{1}{(x-1)^{\frac{1}{3}}}dx=\\
&= \frac{3}{2}+\lim_{b\to\infty}\left[\frac{3}{2}(x-1)^{\frac{2}{3}}\right]_{2}^{b}=\\
&= \frac{3}{2}+\frac{3}{2}\lim_{b\to\infty}\left[(b-1)^{\frac{2}{3}}-(2-1)^{\frac{2}{3}}\right]=\\
&= \frac{3}{2}+\frac{3}{2}\lim_{b\to\infty}\left[(b-1)^{\frac{2}{3}}\right]-\frac{3}{2}=\\
&= \frac{3}{2}\lim_{b\to\infty}\left[(b-1)^{\frac{2}{3}}\right]=\infty
\end{aligned}$$

$$\begin{aligned}
4. \int_0^1 \frac{e^x}{(e^x-1)^{\frac{1}{2}}}dx &= \lim_{a\to 0^+}\int_a^1 \frac{e^x}{(e^x-1)^{\frac{1}{2}}}dx = \\
&= \lim_{a\to 0^+}\left[2\left(e^x-1\right)^{\frac{1}{2}}\right]_a^1 = \\
&= 2\lim_{a\to 0^+}\left[\left(e^1-1\right)^{\frac{1}{2}}-\left(e^a-1\right)^{\frac{1}{2}}\right] = \\
&= 2\left(e^1-1\right)^{\frac{1}{2}} - 2\lim_{a\to 0^+}\left[\left(e^a-1\right)^{\frac{1}{2}}\right] = \\
&= 2\left(e-1\right)^{\frac{1}{2}}
\end{aligned}$$

2.8.3. Lorentz curves and the Gini index.

Area also plays an important role in the study of ***Lorentz curves***, a device used by both economists and sociologists to measure the percentage of a society's wealth that is possessed by a given percentage of its people. To be more specific, the Lorentz curve for a particular society's economy is the graph of the function $L(x)$, which denotes the fraction of total annual national income earned by the lowest-paid $100x\%$ of the wage-earners in the society, for $0 \le x \le 1$. For instance, if the lowest-paid 30% of all wage-earners receive 23% of the society's total income, then $L(0.3) = 0.23$.
The Curve has the basic property where $0 \le L(x) \le 1$ because $L(x)$ is a percentage and $L(1) = 1$ while $L(0) = 0$. Another important property is $L(x) \le x$ because the lowest-paid $100x\%$ of wage-earners cannot receive more than $100x\%$ of total income. When $L(x) = x$, we have total equality (wage-earners with the lowest $100x\%$ of income receive $100x\%$ of the society's wealth).
The Lorentz curve defines a very important economic index called ***the Gini index***. Formally, take the area between $y = x$ and $L(x)$ (this area measures how far a society is from perfect equality), and divide it by the area below the line $y = x$. The ratio is called *the Gini index*. That is,

$$\mathrm{GI} = \frac{\int_0^1 (x - L(x))\,dx}{\frac{1}{2}} = 2\int_0^1 (x - L(x))\,dx.$$

The smaller the index is, the higher the equality in the society it represents.

EXERCISE 2.47. A governmental agency determines that the Lorentz curves for the distribution of income for dentists and contractors in a certain state are given by the functions $L_1(x) = x^{1.7}$ and $L_2(x) = 0.8x^2 + 0.2x$ respectively. For which profession is the distribution of income more fairly distributed?

Solution. The respective Gini indices are

$$\begin{aligned}
G_1 &= 2\int_0^1 (x - L_1(x))\,dx = 2\int_0^1 \left(x - x^{1.7}\right)dx = 2\left(\frac{x^2}{2} - \frac{x^{2.7}}{2.7}\right)_0^1 = 0.2593, \\
G_2 &= 2\int_0^1 (x - L_2(x))\,dx = 2\int_0^1 \left(x - 0.8x^2 - 0.2x\right)dx = 2\left(0.8\frac{x^2}{2} - 0.8\frac{x^3}{3}\right)_0^1 = 0.2667.
\end{aligned}$$

Since the Gini index for dentists is smaller, it follows that in this state, the incomes of dentists are more evenly distributed than those of contractors.

2.9. Taylor series

Need another example to understand why derivatives are so important? one example is the use of the first derivative to approximate the function in a small interval.
Let f be a differentiable function and fix a value x_0 such that $f(x_0) = y_0$. We want to know $y = f(x)$ when x is close to x_0. Denote $dx = \Delta x = x - x_0$ and $dy = \Delta y = y - y_0$. When $dx \ll 1$ is small enough, we can use the following approximation:

$$dy \approx f'(x_0)\,dx.$$

dy is referred to as the *differential* of f at x_0. In other words,

$$y = f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

This approximation is called - *first-order approximation*. The name comes from the fact that we use only the first-order derivative to approximate f in x_0.

EXERCISE 2.48. Approximate $(65)^{1/3}$ using first-order derivative approximation.

Solution. Define $f(x) = x^{1/3}$.Note that $f(64) = 4$, and $f'(x) = \frac{1}{3x^{2/3}}$. Thus, $f'(64) = \frac{1}{3 \cdot 4^2} = \frac{1}{48}$, and

$$65^{1/3} \approx 4 + \frac{1}{48} = 4.020833,$$

when $65^{1/3} = 4.020726$.

EXAMPLE 2.1. Consider a production function $F(x) = \frac{1}{2}\sqrt{x}$. Suppose that the firm is currently using 100 units of labor input x, so that its output is $F(100) = 5$. Now, assume we want to know how much additional output can be achieved by adding one more unit of labor, i.e., we want to compute the marginal product of labor. One way is by computing $F(101) - F(100) = 0.02494...$, which is a bit more complicated to compute. But, there is another way, with the derivative of F in $x = 100$.

$$F'(x) = \frac{1}{4\sqrt{x}} \quad \Rightarrow \quad F'(100) = \frac{1}{40} = 0.025 \approx 0.02494.$$

We got a good approximation of the marginal contribution of one more unit of labor.

EXERCISE 2.49. Consider a firm with a demand function $D(x) = \frac{1}{2}\sqrt{x}$. Estimate the change in output when the firm cuts its labor force from 900 to 896 using first-order derivative approximation.

Solution. Let us begin by using the notations we previously established. $x_0 = 900$, $y_0 = D(900) = 15$, $x = 896$, and $y = D(896)$. Denote $\Delta x = x - x_0$ and $\Delta y = y - y_0$. Thus,

$$\begin{aligned}
\Delta y &\approx D'(x_0)\Delta x \\
&= \frac{1}{4\sqrt{x_0}} \cdot (-4) \\
&= -\frac{1}{30}, \\
y &\approx 15 - \frac{1}{30} = 14.96667,
\end{aligned}$$

while $y = 14.96663$.

EXERCISE 2.50. Consider the cost function $C(x) = 2x^2 + 6x + 12$ for manufacturing x units. Use first-order approximation to approximate the cost of producing the 21^{st} unit. Compare this with the actual cost.

Solution. Note $x_0 = 20,\ C(x_0) = 932,\ x = 21$. Clearly, $C'(x) = 4x + 6$, thus

$$\begin{aligned}
\Delta C &= C(x) - C(x_0) \\
&\approx C'(x_0)\Delta x \\
&= 4 \cdot 20 + 6 = 86,
\end{aligned}$$

while $C(21) - C(20) = 1020 - 932 = 88$.

EXERCISE 2.51. Consider the cost function $C(x) = x^3 - x^2 + 300x + 100$. Use first-order approximation to approximate the effect on the total cost when increasing the production level from 6 to 6.1.

Solution. Denote $x_0 = 6,\ C(x_0) = 2080,\ x = 6.1$. Clearly, $C'(x) = 3x^2 - 2x + 300$, thus

$$\begin{aligned}
\Delta C &\approx C'(x_0)\Delta x \\
&= (396) \cdot 0.1 = 39.6.
\end{aligned}$$

EXERCISE 2.52. In t years, the population of Gotham will be $F(t) = 40 - \frac{8}{t+2}$. Use first-order approximation to approximate the increase during the next 6 months.

Solution. A direct computation shows that $F'(t) = \frac{8}{(t+2)^2}$ when we need to remember that t is taken in years. Hence, $F'(0) = 2$ and the population increases in the next half a year by approximately $F'(0) \cdot 0.5 = 1$.

2.9.1. Taylor approximation. A Taylor expansion of a function f is a method of taking a complex function and approximating its value in a certain set through its derivatives. Let $f : [a,b] \to \mathbb{R}$ be an infinitely differentiable function at $x_0 \in [a,b]$. Its Taylor series $T_f(x)$ in x_0 is defined by

$$T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n .$$

The next element we define is the Taylor polynomial T_N of degree N of the function f by

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n .$$

The polynomial $T_N(x)$ is similar to the Taylor series of f when the sum goes to N instead of ∞. Define the Taylor reminder of degree N by $R_N(x) = f(x) - T_N(x)$. So, if we could bound $R_N(x)$ is some way, we could calculate $f(x)$ through $T_N(x)$, which is a polynomial and therefore, easily computed. For that purpose, we have the Lagrange Reminder Theorem.

THEOREM 2.10. *Fix an interval $I = [a,b]$ such that $x_0 \in I$. Let $f : \mathbb{R} \to \mathbb{R}$ be $N+1$ times differentiable on the open interval (a,b) with $f^{(N)}(x)$ continuous on the closed interval. Then*

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x - x_0)^{N+1}$$

for some real number $c \in (a,b)$.

2.9.2. Taylor series of basic functions.

There are a few basic functions whose Taylor series around $x_0 = 0$ are usually-used and therefore, are worth remembering.

$$\begin{aligned}
1.\ e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \forall x \in \mathbb{R} \\
2.\ \sin(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R} \\
3.\ \cos(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad \forall x \in \mathbb{R} \\
4.\ \ln(x+1) &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} \quad \forall x \in (-1,1] \\
5.\ \arctan(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} \quad \forall x \in [-1,1] \\
6.\ \frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \quad \forall x \in (-1,1).
\end{aligned}$$

2.9.3. A few more words about Taylor series. The first thing you should remember about Taylor series is the definition of the Taylor series, The Taylor polynomial, and the Taylor reminder:

$$\begin{aligned} T_f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \\ T_N(x) &= \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \\ R_N(x) &= f(x) - T_N(x). \end{aligned}$$

These are the definitions. Note that none of them are the same as the function $f(x)$ itself!!! That is, the Taylor series $T_f(x)$ is an infinite sum that depends on the variable x, and in some cases when we plug in specific values of x we get the same value as $f(x)$. This does not mean that its the same representation. In cases that $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$, it means that the value of the function $f(x)$ in x, equals the value we get when we plug the same x in $T_f(x)$, which is an infinite sum. You can think of the Taylor series as a different way to represent the function, that hold for any x in $(x_0 - R, x_0 + R)$.
We defined the Taylor polynomial and the Reminder because we cannot actually compute the infinite sum. What we can do is approximate it by using a finite sum. This is the Taylor polynomial $T_N(x)$. How good is the approximation? Well, for that we have the reminder $R_N(x)$. The reminder gives us the error in our assessment. The theorem we saw states that

$$R_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}(x-x_0)^{N+1}.$$

In words, the reminder depends on the $N+1$ derivative $f^{(N+1)}(c)$ taken at a point $c \in [a,b]$ where the interval $[a,b]$ contains x, x_0.

2.9.4. Exercises.

EXERCISE 2.53. Find the Taylor expansion of $f(x) = x^2$ around $x_0 = 0$. Prove that the function equals the Taylor expansion in every $x \in \mathbb{R}$.

Solution. In order to find the Taylor series of $f(x) = x^2$ we need to compute its derivatives.

$$\begin{aligned} f^{(0)}(x) = x^2 &\Rightarrow f^{(0)}(x_0) = f^{(0)}(0) = 0. \\ f^{(1)}(x) = 2x &\Rightarrow f^{(1)}(x_0) = f^{(1)}(0) = 0. \\ f^{(2)}(x) = 2 &\Rightarrow f^{(2)}(x_0) = f^{(2)}(0) = 2. \\ f^{(3)}(x) = 0 &\Rightarrow f^{(3)}(x_0) = f^{(3)}(0) = 0. \\ &\vdots \\ f^{(n)}(x) = 0 &\Rightarrow f^{(n)}(x_0) = f^{(n)}(0) = 0. \end{aligned}$$

Thus, the Taylor series is

$$\begin{aligned} T_f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n \\ &= 0 + 0 + \frac{f^{(2)}(0)}{2!}x^2 + 0 + \dots \\ &= \frac{2}{2!}x^2 = x^2. \end{aligned}$$

We can see that sum is finite and always well-defined and the reminder is zero as

$$\begin{aligned} R_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1} \\ &= \frac{0}{(N+1)!}\cdot(x-0)^{N+1} = 0, \end{aligned}$$

when $N+1 \geq 3$. Thus,

$$f(x) = x^2 = T_f(x).$$

EXERCISE 2.54. Find the Taylor expansion of $f(x) = x^2$ around $x_0 = 1$. Prove that the function equals the Taylor expansion in every $x \in \mathbb{R}$.

Solution. In order to find the Taylor series of $f(x) = x^2$ we need to compute its derivatives.

$$\begin{aligned} f^{(0)}(x) = x^2 &\Rightarrow f^{(0)}(x_0) = f^{(0)}(1) = 1. \\ f^{(1)}(x) = 2x &\Rightarrow f^{(1)}(x_0) = f^{(1)}(1) = 2. \\ f^{(2)}(x) = 2 &\Rightarrow f^{(2)}(x_0) = f^{(2)}(1) = 2. \\ f^{(3)}(x) = 0 &\Rightarrow f^{(3)}(x_0) = f^{(3)}(1) = 0. \\ &\vdots \\ f^{(n)}(x) = 0 &\Rightarrow f^{(n)}(x_0) = f^{(n)}(1) = 0. \end{aligned}$$

Thus, the Taylor series is

$$\begin{aligned} T_f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f^{(1)}(1)}{1!}(x-1)^1 + \frac{f^{(2)}(1)}{2!}(x-1)^2 + 0 + \dots \\ &= \frac{1}{0!}(x-1)^0 + \frac{2}{1!}(x-1)^1 + \frac{2}{2!}(x-1)^2 \\ &= 1 + 2(x-1) + (x-1)^2 \end{aligned}$$

We can see that sum is finite and always well-defined and the reminder is zero as

$$\begin{aligned} R_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1} \\ &= \frac{0}{(N+1)!}\cdot(x-0)^{N+1} = 0, \end{aligned}$$

when $N+1 \geq 3$. Thus,

$$f(x) = x^2 = 1 + 2(x-1) + (x-1)^2 = T_f(x).$$

EXERCISE 2.55. Find the Taylor expansion of $f(x) = e^x$ around $x_0 = 0$. Prove that the function equals the Taylor expansion in every $x \in \mathbb{R}$.

Solution. In order to find the Taylor series of $f(x) = e^x$ we need to compute its derivatives.

$$\begin{aligned} f^{(0)}(x) = e^x &\Rightarrow f^{(0)}(x_0) = f^{(0)}(0) = 1. \\ f^{(1)}(x) = e^x &\Rightarrow f^{(1)}(x_0) = f^{(1)}(0) = 1. \\ &\vdots \\ f^{(n)}(x) = e^x &\Rightarrow f^{(n)}(x_0) = f^{(n)}(0) = 1 \\ &\vdots \end{aligned}$$

Thus, the Taylor series is

$$\begin{aligned} T_f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-x_0)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}x^n \end{aligned}$$

We can see that sum well-defined for every x, because $a_n = \frac{1}{n!}$ and

$$\begin{aligned} L &= \lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{x\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| \\ &= \lim_{x\to\infty}\frac{n!}{(n+1)!} \\ &= \lim_{x\to\infty}\frac{1}{(n+1)} = 0, \end{aligned}$$

Thus, $R = \infty$. The reminder is zero since

$$\begin{aligned} R_N(x) &= \frac{f^{(N+1)}(c)}{(N+1)!}(x-0)^{N+1} \\ &= \frac{e^c x^{N+1}}{(N+1)!} \to 0 \text{ as } N \to \infty. \end{aligned}$$

Thus,

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n = T_f(x).$$

EXERCISE 2.56. Compute $\sqrt{e}$ with an error of no more than 10^{-5}.

Solution. Define $f(x) = e^x$. We use the Taylor approximation around $x_0 = 0$. Consider the interval $I = [0,1]$ when $x = 0.5 \in I$. The Taylor reminder is $R_N\left(\frac{1}{2}\right) = \frac{f^{(N+1)}(c)}{(N+1)!}\left(\frac{1}{2}-0\right)^{N+1}$ when $c \in (0,1)$. Therefore,

$$R_N\left(\frac{1}{2}\right) = \frac{e^c}{(N+1)!}\left(\frac{1}{2}\right)^{N+1} < \frac{3}{(N+1)!2^{N+1}}.$$

We require that $R_N\left(\frac{1}{2}\right) < 10^{-5}$, which occurs when $N \geq 6$. Thus we can take $N = 6$ and get

$$\sqrt{e} \approx \sum_{n=0}^{6} \frac{1}{n!}\left(\frac{1}{2}\right)^n.$$

EXERCISE 2.57. Compute the Taylor polynomial $T_3(x)$ for the function $f(x) = x^{1/3}$ around $x_0 = 27$. Compute $27.1^{1/3}$ using the polynomial and give an upper bound on the Taylor reminder.

Solution. We need to compute the derivatives of the function, up to the forth order.

$$\begin{aligned} f^{(0)}(x) &= x^{1/3} &&\Rightarrow && f^{(0)}(27) = 3; \\ f^{(1)}(x) &= \frac{1}{3x^{2/3}} &&\Rightarrow && f^{(1)}(27) = \frac{1}{27}; \\ f^{(2)}(x) &= \frac{-2}{3^2x^{5/3}} &&\Rightarrow && f^{(2)}(27) = -\frac{2}{3^7}; \\ f^{(3)}(x) &= \frac{10}{3^3x^{8/3}} &&\Rightarrow && f^{(3)}(27) = \frac{10}{3^{11}}; \\ f^{(4)}(x) &= -\frac{80}{3^4x^{11/3}} &&\Rightarrow && f^{(4)}(27) = -\frac{80}{3^{15}}. \end{aligned}$$

Therefore,

$$\begin{aligned} T_3(x) &= \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}(x-27)^n = \\ &= \frac{3}{0!}(x-27)^0 + \frac{1}{27}(x-27)^1 - \frac{1}{3^7}(x-27)^2 + \frac{5}{3^{12}}(x-27)^3 = \\ &= 3 + \frac{1}{27}(x-27) - \frac{1}{3^7}(x-27)^2 + \frac{5}{3^{12}}(x-27)^3 . \end{aligned}$$

And,

$$\begin{aligned} T_3(27.1) &= 3 + \frac{1}{27}(0.1) - \frac{1}{3^7}(0.1)^2 + \frac{5}{3^{12}}(0.1)^3 = \\ &= 3 + \frac{1}{270} - \frac{1}{3^7 \cdot 100} + \frac{1}{3^{12} \cdot 200}, \end{aligned}$$

with a reminder of

$$\begin{aligned} |R_3(27.1)| &= \left| \frac{f^{(4)}(c)}{4!}(27.1-27)^4 \right| = \\ &= \frac{80}{81 \cdot 4! c^{\frac{11}{3}}} \cdot \frac{1}{10000} \leq \\ &\leq \frac{1}{24 \cdot 3^{11} \cdot 10000} \backsimeq 2.35 \cdot 10^{-11}, \end{aligned}$$

when $c \in [27, 27.1]$.

2.9.5. Convergence of power series.

Clearly, $f(x) = T_f(x)$ if and only if $\lim_{N\to\infty} R_N(x) = 0$. However, we should point out that $T_f(x)$ does not always converge, meaning that $T_f(x)$ may not even be well-defined in certain parts of $\mathbb{R}$. Thus, we have a few methods of assuring of knowing when $T_f(x)$converges.

Consider s Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$. Let R be a non-negative real number, such that for every $x \in (x_0 - R, x_0 + R)$ the series converges. In order to compute R, known as the *radius of convergence*, we can use the two following methods:

(1) The root formula that states that $R = \frac{1}{L}$ and $L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$, where $a_n = \frac{f^{(n)}(x_0)}{n!}$.
(2) The ratio formula that states that$R = \frac{1}{L}$ and $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$, where $a_n = \frac{f^{(n)}(x_0)}{n!}$.

Exercise 2.58. Compute the Taylor series of $f(x) = \frac{1}{2x+3}$ around $x_0 = 1$. Find its radius of convergence.

Solution. Note that

$$
\begin{aligned}
f(x) &= \frac{1}{2x+3}\\
&= \frac{1}{2x-2+2+3} =\\
&= \frac{1}{2(x-1)+5} =\\
&= \frac{1}{5}\cdot\frac{1}{1+\frac{2}{5}(x-1)} =\\
&\overset{t=-\frac{2}{5}(x-1)}{=} \frac{1}{5}\cdot\frac{1}{1-t}\\
&= \frac{1}{5}\sum_{n=0}^{\infty} t^n =\\
&= \frac{1}{5}\sum_{n=0}^{\infty}\frac{(-2)^n}{5^n}(x-1)^n\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{5^{n+1}}(x-1)^n .
\end{aligned}
$$

when we used the formula of the geometric series. We know that the geometric series $\sum_{n=0}^{\infty} t^n$ converges in every $t\in(-1,1)$, thus

$$
\begin{array}{ccc}
-1< & t & <1\\
-1< & -\frac{2(x-1)}{5} & <1\\
-\dfrac{5}{2}< & x-1 & <\dfrac{5}{2}\\
-\dfrac{3}{2}< & x & <\dfrac{7}{2},
\end{array}
$$

that is $x_0-R=-\frac{3}{2}$ and $x_0+R=\frac{7}{2}$. Thus, the radius of convergence is $R=\frac{5}{2}$.

Remark 2.8. The differentiation and integration of a Taylor series is done just as any other polynomial of a finite degree. That is, we differentiate and integrate point-wise, such that$\left[a_n(x-x_0)^n\right]'=na_n(x-x_0)^{n-1}$ for every term of the series.

Exercise 2.59. Prove using Taylor series that $\lim_{x\to0}\frac{\sin(x)}{x}=1$.

Solution. Plug-in the Taylor series and compute the limit term-by-term to get

$$
\begin{aligned}
\lim_{x\to0}\frac{\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}}{x} &= \lim_{x\to0}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n}\\
&= \lim_{x\to0}\left[\frac{(-1)^0}{(0+1)!}x^0+\frac{(-1)^1}{(2+1)!}x^2+\frac{(-1)^2}{(4+1)!}x^4+\cdots\right]\\
&= \lim_{x\to0}\left[1+0+0+\cdots\right]\\
&= 1.
\end{aligned}
$$

Exercise 2.60. Expand each of the following functions as a Taylor series around x_0 and find its interval of convergence.

(1) $f(x)=\frac{1}{4+3x},\quad x_0=0;$
(2) $f(x)=\frac{1}{16+2x^3},\quad x_0=0;$
(3) $f(x)=\frac{x-2}{1-x},\quad x_0=2;$
(4) $f(x)=\frac{2x}{(x^2+1)^2},\quad x_0=0.$

Solution.

(1) We wish to use the Taylor series of $\frac{1}{1-x}$, so

$$f(x) = \frac{1}{4} \cdot \frac{1}{1+\frac{3}{4}x} \overset{t=-\frac{3}{4}x}{=} \frac{1}{4} \cdot \frac{1}{1-t} = \frac{1}{4}\sum_{t=0}^{\infty} t^n = \sum_{t=0}^{\infty} \frac{(-3)^n}{4^{n+1}} x^n.$$

We know that the geometric series $\sum_{n=0}^{\infty} t^n$ converges in every $t \in (-1,1)$, thus

$$\begin{aligned} -1 < \quad & t & < 1 \\ -1 < \quad & -\tfrac{3}{4}x & < 1 \\ -\frac{4}{3} < \quad & x & < \frac{4}{3}, \end{aligned}$$

and the interval of convergence is $\left[-\frac{4}{3}, \frac{4}{3}\right]$.

(2) Again, doing the same transitions as in the previous exercise

$$f(x) = \frac{1}{16} \cdot \frac{1}{1+\left(\frac{x}{2}\right)^3} \overset{t=-\left(\frac{x}{2}\right)^3}{=} \frac{1}{16}\sum_{t=0}^{\infty} \left[-\left(\frac{x}{2}\right)^3\right]^n = \sum_{t=0}^{\infty} \frac{(-1)^n}{2^{3n+4}} x^{3n}.$$

We know that the geometric series $\sum_{n=0}^{\infty} t^n$ converges in every $t \in (-1,1)$, thus

$$\begin{aligned} -1 < \quad & t & < 1 \\ -1 < \quad & -\left(\tfrac{x}{2}\right)^3 & < 1 \\ -2 < \quad & x & < 2, \end{aligned}$$

and the interval of convergence is $[-2,2]$.

(3) We wish to use the same formula as before, however now we need to make sure that $t = x-2$, such that we get the correct value for x_0. In other words, we want to reach a term that includes $\frac{1}{1\pm(x-2)}$

$$\begin{aligned} \frac{1}{1-x} &= \frac{1}{1-(x-2+2)} = \frac{1}{-1-(x-2)} = -\frac{1}{1+(x-2)} \\ &\overset{t=-(x-2)}{=} -\frac{1}{1-t} = -\sum_{t=0}^{\infty} (-1)^n (x-2)^n = \sum_{t=0}^{\infty} (-1)^{n+1} (x-2)^n. \end{aligned}$$

We know that the geometric series $\sum_{n=0}^{\infty} t^n$ converges in every $t \in (-1,1)$, thus

$$\begin{aligned} -1 < \quad & t & < 1 \\ -1 < \quad & -x+2 & < 1 \\ 1 < \quad & x & < 3, \end{aligned}$$

and the interval of convergence is $[1,3]$.

(4) First, we have

$$\frac{1}{1+x^2} \overset{t=-x^2}{=} \sum_{t=0}^{\infty} t^n = \sum_{t=0}^{\infty} (-1)^n x^{2n}.$$

The interval of convergence is $[-1,1]$. Now we can differentiate and get

$$\left[\frac{1}{1+x^2}\right]' = -\frac{2x}{(1+x^2)^2} = \sum_{t=1}^{\infty} 2n(-1)^n x^{2n-1},$$

and $f(x) = \sum_{t=1}^{\infty} \left[(-1)^{n+1} 2n \cdot x^{2n-1}\right]$.

Part 2

Linear algebra

CHAPTER 3

Linear models and matrix algebra

3.1. Examples of linear models in economics

3.1.1. Tax benefits for charitable contributions.

We start with a basic example of a linear model in economics, which exemplify the need to know linear algebra.[1]

EXAMPLE 3.1. Assume the a company with a before-tax annual profits of \$100,000 decides to contribute 10% of its after-tax profits to the Red Cross. It must pay 5% state tax (after the donation), and a federal tax of 40% (after the state tax and donation).

EXERCISE 3.1. How much does the company pay in state taxes, federal taxes, and the Red-Cross donation?

Solution. First define the variables C, S, and F as the charity contribution, the state tax, and the federal tax, respectively. We need to write down all the relevant relations between the different variables. The after-tax profits are $100,000 - S - F$, so

$$\begin{aligned} C &= 0.1\,(100,000 - S - F) \\ C + 0.1S + 0.1F &= 10,000. \end{aligned}$$

The fact that the state tax is 5% taken from the net profit (without the donation) means that

$$\begin{aligned} S &= 0.05\,(100,000 - C) \\ S + 0.05C &= 5,000. \end{aligned}$$

The federal tax is 40% for the profit, without the donation and state tax. Hence,

$$\begin{aligned} F &= 0.4\,(100,000 - C - S) \\ F + 0.4C + 0.4S &= 40,000. \end{aligned}$$

We can see that we got a system of three equations

$$\begin{aligned} C + 0.1S + 0.1F &= 10,000 \\ 0.05C + S + 0 \cdot F &= 5,000 \\ 0.4C + 0.4S + F &= 40,000, \end{aligned}$$

that we can solve plunging-in one relation into the others. The results are $C = 5,956$, $S = 4,702$, and $F = 35,737$.

EXERCISE 3.2. Assuming that the company decided not to make any contribution to the Red-Cross. Find the net cost of its \$5,956 contribution.

Solution. We need to write down the equations again, when now the value of C is zero. Therefore,

$$\begin{aligned} S &= 5,000 \\ 0.4S + F &= 40,000, \end{aligned}$$

[1]These examples are taken from the book *Mathematics for Economists* by Carl P. Simon and Lawrence Blume.

hence , $S = 5,000$ and $F = 40,000 - 0.4 \cdot 5,000 = 38,000$. The net profit is $100,000 - 38,000 - 5000 = 57,000$ instead of $100,000 - 5,956 - 4,702 - 35,737 = 53,605$, and the contribution of \$5,956 cost was

$$\$57,000 - \$53,605 = \$3,395.$$

This means that 57% of the contribution came from the company and 43% came from the state \ other tax-payers.

Exercise 3.3. In New-York federal income taxes are deducted from the state taxes (meaning, you do not pay a state tax on the federal taxes). Solve the previous example in case the company is based in New-York.

Solution. Using the previously-defined notations C, F, and S, the new system of linear equations is updated according to the new equation $S = 0.05\,(100,000 - C - F)$, which implies

$$\begin{aligned} C + 0.1S + 0.1F &= 10,000 \\ 0.05C + S + 0.05F &= 5,000 \\ 0.4C + 0.4S + F &= 40,000. \end{aligned}$$

The solution for this system is $C = 6,070,\ S = 2,875$, and $F = 36,422$.

3.1.2. The Leontief model.

There are many other economic models that yield systems of linear equations, sometimes more complicated to analyze than the previous example. In such cases, finding an easy way to solve systems of linear equations becomes necessary, and this is another reason for us to study *Linear Algebra.* We continue with another economic example, which is the *Lenotief linear-production model.* The Leontief model, named after Wassily Leontief who won the Nobel prize in economics in 1973, is a model for the economics of a whole country or region. The model has two variations, one for closed economies and one for open economies. We deal with each separately.

- **The closed Leontief model.** Consider an economy with n sectors. Each sector i produces x_i units of a single homogeneous good. Assume that the j^{th} sector, in order to produce 1 unit, must use a_{ij} units from sector i. Furthermore, assume that each sector sells some of its output to other sectors. Then we might write

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n,$$

  and the same equations could be written for each good i. In simple terms, if an economy needs to produce x_j units of good j, it will need $a_{ij}x_j$ units of good i. Now we have a system a linear equations, and its solution dictates the amount that every sector needs to produce in equilibrium, such that the supply will equal the demand.
- **The open Leontief model.** In addition to the closed model, assume that each sector sells some of its output to other sectors (intermediate output) and some of its output to consumers (final output, or final demand). Call final demand in the i^{th} sector d_i. Then we might write

$$x_i = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n + d_i,$$

  or total output equals intermediate output plus final output. Again we can write the same equation for every good i, and the solution will dictate the amount of goods each sector needs to produce such that the market balances.

Exercise 3.4. The economy of a country produces only grapes and wine. The production of 1 kg of grapes requires 0.5 kg of grapes, 1 worker, and no wine. The production of 1 liter of wine requires 0.5 kg grapes, 1 worker, and 0.25 liter of wine. The country has 10 workers that demand 1 kg of grapes and 3 liters of wine, overall. Write the relevant input-output system and solve it.

Solution. We need to write down the relevant equations. Let x_1 and x_2 be the produced amounts of grapes and wine, respectively. Since we do not produce workers, we need to verify, eventually, that there are enough worker to produce the required amounts. The system of equations is

$$\begin{aligned} x_1 &= 0.5x_1 + 0.5x_2 + 1, \\ x_2 &= 0x_1 + 0.25x_2 + 3. \end{aligned}$$

You can see the demand on the right-hand side (RHS) and the supply on the left-hand side (LHS). Hence,

$$x_1 = 6, \ x_2 = 4.$$

We can see that the numbers of workers needed is exactly 10.

EXERCISE 3.5. Suppose now that the production of grapes requires 7/8 liter of wine. Write down the updated system and solve it.

Solution. The new system of equations is

$$\begin{aligned} x_1 &= 0.5x_1 + 0.5x_2 + 1, \\ x_2 &= 0.875x_1 + 0.25x_2 + 3. \end{aligned}$$

And the solution is

$$\begin{aligned} x_1 &= x_2 + 2 \\ 6x_2 &= 7x_1 + 24 \\ &\Downarrow \\ x_1 &= x_2 + 2 \\ 6x_2 &= 7(x_2 + 2) + 24 \\ &\Downarrow \\ x_2 &= -38, \ x_1 = -36, \end{aligned}$$

which is clearly infeasible.

In general we want to know how many solutions are there (if any)? and we wish to find them. The basic ways to solve these systems, that we already know from high school, are substitution and elimination of variables. In other words, we use algebraic manipulation to find the value of each variable, one by one, and by doing so, we solve the equation. Consider for example the following three-good economy summarized in the following table:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 0 & 0.4 & 0.3 \\ 0.2 & 0.12 & 0.14 \\ 0.5 & 0.2 & 0.05 \end{pmatrix}$$

Remember that, for example, $a_{21} = 0.2$ is the amount of units of good 2, it takes to produce one unit of good 1 (along with $a_{11} = 0$ units of good 1 and $a_{31} = 0.5$ units of good 3). Now assume that there is an exogenous demand for 130 units of good 1 (that is, $d_1 = 130$), 74 units of good 2 (that is,$d_2 = 74$), and 95 units of good 3 ($d_3 = 95$). We can write down the system of linear equations as follows:

$$\begin{aligned} x_1 &= 0x_1 + 0.4x_2 + 0.3x_3 + 130 \\ x_2 &= 0.2x_1 + 0.12x_2 + 0.14x_3 + 74 \\ x_3 &= 0.5x_1 + 0.2x_2 + 0.05x_3 + 95 \end{aligned}$$

and this is equivalent to

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ -0.2x_1 + 0.88x_2 - 0.14x_3 &= 74 \\ -0.5x_1 - 0.2x_2 + 0.95x_3 &= 95. \end{aligned}$$

We now try to solve these equations by using a few basic operations:

- multiplying both side of an equation with a non-zero, real number;
- add a multiple of one equation to another equation;
- interchanging the order of equations.

These three operations are called *elementary equation operations* and they are all reversible. Therefore, we can use them to solve the system above. First thing we can do is to add 0.2 times of the first equation to the second, such that x_1 is eliminated from the second equation, and add 0.5 times of the first equation to the third equation. Then we get

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ 0.8x_2 - 0.2x_3 &= 100 \\ -0.4x_2 + 0.8x_3 &= 160. \end{aligned}$$

Now we can add 0.5 times the second equation to the third and get

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 \\ 0.8x_2 - 0.2x_3 &= 100 \\ 0.7x_3 &= 210. \end{aligned}$$

We can see that $x_3 = 300$, which means that $x_2 = 200$ and $x_1 = 300$. Note that we wasted some time on rewriting the name of the variables and the signs over and over again and therefore a more efficient method is needed. In addition, though these techniques are simple, they might not be sufficient when the number of variables is high. For these reasons, we have *the matrix methods.*

3.1.3. Matrix methods.

Let us consider the following system of linear equations that is quite similar to a system describing an open Leontief model,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \quad & \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned} \tag{3.1.1}$$

The system given in 3.1.1 could be abbreviated using the following table

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix},$$

when we omit the plus signs, the names of the variables, and the equality signs. We can preform the same elementary operations on this table, and reach a similar result as we did before. That is, assume that the table is

$$\begin{pmatrix} 1 & -0.4 & -0.3 & 130 \\ -0.2 & 0.88 & -0.14 & 74 \\ -0.5 & -0.2 & 0.95 & 95 \end{pmatrix},$$

just as we had with the previous input-output model. We can use the three elementary operations we discussed and reach the same result as we did before. This table is called *the augmented matrix,* and a similar table without the b_i column is called *the coefficient matrix.* The term *matrix* relates to every such table that includes numbers. The process of eliminating variables from equations (transforming coefficients to zero) is called *The Gauss-Jordan elimination* process. Its optimal final result is a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{pmatrix},$$

which gives us the solution for the system of linear equations. It is done by the three basic operation of:

(1) Multiplying a row with a non-zero, real number;
(2) Add a multiple of one row to another row;
(3) Interchanging the order of rows.

Our main goal is to reach a *row echelon form*, defined as follows.

DEFINITION 3.1. A row of a matrix is said to have k *leading zeros* if the first k elements of the row are all zeros and the $k+1$ element of the row is not zero. A matrix is in *row echelon form* if each row has more leading zeros than the previous row.

EXAMPLE 3.2. Let us solve the following system by the Gauss-Jordan elimination process.

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x - z &= 0 \end{aligned}$$

Solution. We begin by writing down the augmented matrix and explaining every operation we make when row i is denoted by L_i.

$$\begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 2 & 3 & 7 & -2 & 9 \\ 3 & 5 & 13 & -9 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{L_2 - 2L_1 \to L_2} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 3 & 5 & 13 & -9 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{L_3 - 3L_1 \to L_3}$$

$$\begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{pmatrix} \xrightarrow{L_4 + 2L_1 \to L_4} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ 0 & 3 & 6 & -5 & 0 \end{pmatrix} \xrightarrow{L_1 - L_2 \to L_1}$$

$$\begin{pmatrix} 1 & 0 & 2 & -4 & -9 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ 0 & 3 & 6 & -5 & 0 \end{pmatrix} \xrightarrow{L_3 - 2L_2 \to L_3} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 3 & 6 & -5 & 0 \end{pmatrix} \xrightarrow{L_4 - 3L_2 \to L_4}$$

$$\begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix} \xrightarrow{\frac{1}{2}L_3 \to L_3} \begin{pmatrix} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 3 & -11 & -27 \end{pmatrix} \xrightarrow{L_4 - 3L_3 \to L_4}$$

$$\begin{pmatrix}1&1&3&-2&0\\0&1&1&2&9\\0&0&1&-\frac{7}{2}&-\frac{17}{2}\\0&0&0&-\frac{1}{2}&-\frac{3}{2}\end{pmatrix}\xrightarrow{-2L_4\to L_4}\begin{pmatrix}1&1&3&-2&0\\0&1&1&2&9\\0&0&1&-\frac{7}{2}&-\frac{17}{2}\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_1+2L_4\to L_1}$$

$$\begin{pmatrix}1&1&3&0&6\\0&1&1&2&9\\0&0&1&-\frac{7}{2}&-\frac{17}{2}\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_2-2L_4\to L_2}\begin{pmatrix}1&1&3&0&6\\0&1&1&0&3\\0&0&1&-\frac{7}{2}&-\frac{17}{2}\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_3+\frac{7}{2}L_4\to L_3}$$

$$\begin{pmatrix}1&1&3&0&6\\0&1&1&0&3\\0&0&1&0&2\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_1-3L_3\to L_1}\begin{pmatrix}1&1&0&0&0\\0&1&1&0&3\\0&0&1&0&2\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_2-L_3\to L_2}$$

$$\begin{pmatrix}1&1&0&0&0\\0&1&0&0&1\\0&0&1&0&2\\0&0&0&1&3\end{pmatrix}\xrightarrow{L_1-L_2\to L_1}\begin{pmatrix}1&0&0&0&-1\\0&1&0&0&1\\0&0&1&0&2\\0&0&0&1&3\end{pmatrix}.$$

And the solution is $w=-1,\ x=2,\ y=2,\ z=3$.

EXERCISE 3.6. Write the augmented matrices and solve the systems with elementary rows operations for the following systems:

(1)

$$\begin{aligned}x_1-x_2+2x_3&=1\\2x_1+2x_3&=1\\x_1-3x_2+4x_3&=2\end{aligned}$$

(2)

$$\begin{aligned}x_1-x_2+2x_3&=1\\2x_1-4x_2+6x_3&=3\\x_1-3x_2+4x_3&=2\end{aligned}$$

(3)

$$\begin{aligned}x_1-x_2+2x_3&=1\\2x_1-4x_2+6x_3&=2\\x_1-3x_2+4x_3&=2\end{aligned}$$

Solution.

(1) The augmented matrix is

$$\begin{pmatrix}1&-1&2&1\\2&0&2&1\\1&-3&4&2\end{pmatrix}.$$

Its solution is

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 2 & 0 & 2 & 1\\ 1 & -3 & 4 & 2\end{pmatrix} \xrightarrow{L_2-2L_1\to L_2} \begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 2 & -2 & -1\\ 1 & -3 & 4 & 2\end{pmatrix} \xrightarrow{L_3-L_1\to L_3}$$

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 2 & -2 & -1\\ 0 & -2 & 2 & 1\end{pmatrix} \xrightarrow{\frac{1}{2}L_2\to L_2} \begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 1 & -1 & -\frac{1}{2}\\ 0 & -2 & 2 & 1\end{pmatrix} \xrightarrow{L_3+2L_2\to L_3}$$

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 1 & -1 & -\frac{1}{2}\\ 0 & 0 & 0 & 0\end{pmatrix} \xrightarrow{L_1+L_2\to L_1} \begin{pmatrix}1 & 0 & 1 & \frac{1}{2}\\ 0 & 1 & -1 & -\frac{1}{2}\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

This means that $x_1 = \frac{1}{2} - x_3$ and $x_2 = -\frac{1}{2} + x_3$, where $x_3 \in \mathbb{R}$.

(2) In the following answer we present only the augmented matrix and final solution.

$$\hat{A} = \begin{pmatrix}1 & -1 & 2 & 1\\ 2 & -4 & 6 & 3\\ 1 & -3 & 4 & 2\end{pmatrix} \quad \Rightarrow \quad x_1 = \frac{1}{2} - x_3,\ x_2 = -\frac{1}{2} + x_3,\ x_3 \in \mathbb{R}.$$

We got the same solution as before.

(3) The augmented matrix is

$$\hat{A} \quad = \quad \begin{pmatrix}1 & -1 & 2 & 1\\ 2 & -4 & 6 & 2\\ 1 & -3 & 4 & 2\end{pmatrix}.$$

By the Gauss-elimination process we get

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 2 & -4 & 6 & 2\\ 1 & -3 & 4 & 2\end{pmatrix} \xrightarrow{L_2-2L_1\to L_2} \begin{pmatrix}1 & -1 & 2 & 1\\ 0 & -2 & 2 & 0\\ 1 & -3 & 4 & 2\end{pmatrix} \xrightarrow{L_3-L_1\to L_3}$$

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 0 & -2 & 2 & 0\\ 0 & -2 & 2 & 1\end{pmatrix} \xrightarrow{-\frac{1}{2}L_2\to L_2} \begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 1 & -1 & 0\\ 0 & -2 & 2 & 1\end{pmatrix} \xrightarrow{L_3+2L_2\to L_3}$$

$$\begin{pmatrix}1 & -1 & 2 & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

We got to a point where we know that a solution does not exists as $0x_1 + 0x_2 + 0x_3 = 1$, and this cannot hold.

Remark 3.1. Henceforth and unless stated otherwise, we will present the full elimination process. However, we will present the relevant matrices and final results.

Exercise 3.7. Use the Gauss-Jordan elimination process to solve

$$\begin{cases} 3x + 3y &= 4\\ -x - y &= 10. \end{cases}$$

What happens?

Solution. By the Gauss-Jordan elimination process we get to the matrix

$$\begin{pmatrix}3 & 3 & 4\\ -1 & -1 & 10\end{pmatrix} \xrightarrow{L_2+\frac{1}{3}L_1\to L_2} \begin{pmatrix}3 & 3 & 4\\ 0 & 0 & 11\frac{1}{3}\end{pmatrix},$$

which means that a solution does not exist as $0 \neq 11\frac{1}{3}$.

EXERCISE 3.8. Solve the system

$$\begin{cases} -4x + 6y + 4z &= 4 \\ 2x - y + z &= 1. \end{cases}$$

Solution. The solution is given by the following set $\{(x, y, z): \ x = \frac{5}{4} - \frac{5}{4}z, \ y = \frac{3}{2} - \frac{3}{2}z, \ z \in \mathbb{R}\}$.

EXERCISE 3.9. Use the Gauss-Jordan elimination process to determine for what values of the parameter k the system

$$\begin{cases} x + y &= 1 \\ x - ky &= 1 \end{cases}$$

has no solutions, one solutions, and more than one solution.

Solution. The augmented matrix and elimination process yield

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -k & 1 \end{pmatrix} \xrightarrow{L_2 - L_1 \to L_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & -k-1 & 0 \end{pmatrix}.$$

If $k = -1$, than we have infinitely many solutions, as $x = 1 - y$ and $y \in \mathbb{R}$. Otherwise,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -k-1 & 0 \end{pmatrix} \xrightarrow{\frac{1}{-k-1}L_2 \to L_2} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\xrightarrow{L_1 - L_2 \to L_1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

and we have a unique solution $x = 1, y = 0$.

EXERCISE 3.10. Which of the following equations are linear?

(1)
$$3x_1 - 4x_2 + 9x_3 = 17.$$

(2)
$$x_1 x_2 x_3 = 5.$$

(3)
$$x^2 + 9y = 6.$$

(4)
$$x_1 + 4^{0.5}x_2 + \sqrt{3}x_3 = 17^2 x_1.$$

(5)
$$(x_1 - x_2)\, x_3 = 5.$$

Solution. Equations 1 and 4 are linear, all other are not.

EXERCISE 3.11. Solve the following systems by the Gauss-Jordan elimination process.

(1)
$$\begin{aligned} x - 3y + 6z &= -1 \\ 2x - 5y + 10z &= 0 \\ 3x - 8y + 17z &= 1; \end{aligned}$$

(2)
$$\begin{aligned} x + y + z &= 0 \\ 12x + 2y - 3z &= 5 \\ 3x + 4y + z &= -4; \end{aligned}$$

(3)
$$\begin{aligned} 3x + 3y &= 4 \\ x - y &= 10; \end{aligned}$$

(4)
$$\begin{aligned} 4x + 2y - 3z &= 1 \\ 6x + 3y - 5z &= 0 \\ x + y + 2z &= 9; \end{aligned}$$

(5)
$$\begin{aligned} 2x + 2y - z &= 2 \\ x + y + z &= -2 \\ 2x - 4y + 3z &= 0. \end{aligned}$$

Solution. The augmented matrices and solutions are

(1)
$$\hat{A} = \begin{pmatrix} 1 & -3 & 6 & -1 \\ 2 & -5 & 10 & 0 \\ 3 & -8 & 17 & 1 \end{pmatrix} \quad \Rightarrow \quad x = 5, \ y = 6, \ z = 2.$$

(2)
$$\hat{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 12 & 2 & -3 & 5 \\ 3 & 4 & 1 & -4 \end{pmatrix} \quad \Rightarrow \quad x = 1, \ y = -2, \ z = 1.$$

(3)
$$\hat{A} = \begin{pmatrix} 3 & 3 & 4 \\ 1 & -1 & 10 \end{pmatrix} \quad \Rightarrow \quad x = \frac{17}{3}, \ y = -\frac{13}{3}.$$

(4)
$$\hat{A} = \begin{pmatrix} 4 & 2 & -3 & 1 \\ 6 & 3 & -5 & 0 \\ 1 & 1 & 2 & 9 \end{pmatrix} \quad \Rightarrow \quad x = 2, \ y = 1, \ z = 3.$$

(5)
$$\hat{A} = \begin{pmatrix} 2 & 2 & -1 & 2 \\ 1 & 1 & 1 & -2 \\ 2 & -4 & 3 & 0 \end{pmatrix} \quad \Rightarrow \quad x = 1, \ y = -1, \ z = -2.$$

EXERCISE 3.12. Use the Gauss-Jordan elimination process to determine for what values of the parameter k the system

$$\begin{cases} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= k \end{cases}$$

has a solution.

Solution. The reduced echelon form of the system is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 8+k \end{pmatrix},$$

which means that for $k=-8$, the system has the unique solution $x=y=1$. If $k\neq -8$, a solution does not exist.

3.2. Matrix algebra

A known fact is that a system of linear equations must have either no solution, one solution, or infinitely many solutions. An important question about these systems is when a solution exists and how many solutions are there? We start with a few basic definitions regarding a systems of linear equations, denoted

$$A\mathbf{x} = \mathbf{b}, \tag{3.2.1}$$

when

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

That is, A is the coefficient matrix, and $\mathbf{x}, \mathbf{b}$ are called *vectors* (these are basically matrices with a single column).[2]

3.2.1. Basic operations.

Before we continue with the problems of solving systems of linear equations, we need to elaborate on matrices. A *matrix* is a rectangular array of numbers, so any table of data is a matrix. Its size is indicated by the number of rows and its number of columns. In other words, a matrix with m rows and n columns is called a "$m\times n$ matrix". The entry (i.e., the number) in the i^{th} row and j^{th} column is called the "(i,j)-th entry".

There are basic operations we can preform on matrices. The basic ones are addition and scalar multiplication.

- *Scalar multiplication.* A scalar is an ordinary real-number. The product of a scalar r and a $m\times n$ matrix A is

$$rA = r\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n} \\ ra_{21} & ra_{22} & \cdots & ra_{2n} \\ \vdots & \vdots & \dots & \vdots \\ ra_{m1} & ra_{m2} & \cdots & ra_{mn} \end{pmatrix}.$$

  In words, the scalar r multiplies every entry in the matrix.

[2] Equality (3.2.1) will be explained later on. In simple terms, it is an equability between two vectors, which means that every coordinate of the vector on the RHS equals every coordinate of the vector on the LHS.

- *Addition.* The addition of two matrices is defined if and only if the two matrices are of the same size. Let A and B be two matrices of size $m \times n$. Then,

$$\begin{aligned} A + B &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}. \end{aligned}$$

Using these two operations, we can define the subtraction $A - B$ as the addition of two matrices, when the latter is multiplied by the scalar (-1) such that $A - B = A + (-B)$.
The next operation is multiplication of matrices. The multiplication of two matrices A, C is defined if and only if the number of columns in A equals then umber of rows in C. Therefore, assume that A is a $m \times n$ matrix and C is an $n \times k$ matrix. The product AC is defined by

$$\begin{aligned} AC &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nk} \end{pmatrix} \\ &= \begin{pmatrix} \sum_{i=1}^{n} a_{1i}c_{i1} & \sum_{i=1}^{n} a_{1i}c_{i2} & \cdots & \sum_{i=1}^{n} a_{1i}c_{ik} \\ \sum_{i=1}^{n} a_{2i}c_{i1} & \sum_{i=1}^{n} a_{2i}c_{i2} & \cdots & \sum_{i=1}^{n} a_{2i}c_{ik} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \sum_{m=1}^{n} a_{im}c_{mj} & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{n} a_{mi}c_{i1} & \sum_{i=1}^{n} a_{mi}c_{i2} & \cdots & \sum_{i=1}^{n} a_{mi}c_{ik} \end{pmatrix}. \end{aligned}$$

Or, in simpler terms, the (i,j)-th entry of AC is $\sum_{m=1}^{n} a_{im}c_{mj}$. That is, we sum the *products* of the entries of the i^{th} row in A with the entries of the j^{th} column in C.

EXERCISE 3.13. Define the matrices

$$A = \begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix}.$$

Compute $A + B$, $A - B$, $3A - 2B$, AB, and BA.

Solution.

$$\begin{aligned} A + B &= \begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 9 & 3 \\ 2 & 7 \end{pmatrix}, \\ A - B &= \begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & -5 \\ 6 & -1 \end{pmatrix}, \\ 3A - 2B &= 3\begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix} - 2\begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 12 & -11 \\ 16 & 1 \end{pmatrix}, \\ AB &= \begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 20 & 20 \\ 6 & 28 \end{pmatrix}, \\ BA &= \begin{pmatrix} 3 & 4 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 6 & -1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 34 & 9 \\ 4 & 14 \end{pmatrix}. \end{aligned}$$

EXERCISE 3.14. Perform the following computations:

$$a.\quad 4\begin{pmatrix}6 & -4 & 2\\ 3 & 3 & 9\end{pmatrix} - 5\begin{pmatrix}1 & 0 & 6\\ 2 & 3 & -5\end{pmatrix}.$$

$$b.\quad \begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix}\begin{pmatrix}3 & -2 & -1\\ 4 & 4 & 1\end{pmatrix}.$$

$$c.\quad \begin{pmatrix}2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\\ 1 & 1 & 0 & 7\end{pmatrix}\begin{pmatrix}2 & 2 & 1\\ 3 & -3 & 0\\ 7 & 2 & -1\\ -4 & -5 & 0\end{pmatrix}.$$

$$d.\quad \begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1 & -1\\ 2 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}\begin{pmatrix}5\\ -1\\ -1\end{pmatrix}.$$

Solution.

$$a.\quad 4\begin{pmatrix}6 & -4 & 2\\ 3 & 3 & 9\end{pmatrix} - 5\begin{pmatrix}1 & 0 & 6\\ 2 & 3 & -5\end{pmatrix} = \begin{pmatrix}19 & -16 & -22\\ 2 & -3 & 61\end{pmatrix}.$$

$$b.\quad \begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix}\begin{pmatrix}3 & -2 & -1\\ 4 & 4 & 1\end{pmatrix} = \begin{pmatrix}10 & 0 & -1\\ 13 & 18 & 5\end{pmatrix}.$$

$$c.\quad \begin{pmatrix}2 & 1 & 0 & 0\\ 3 & 2 & 1 & 0\\ 1 & 1 & 0 & 7\end{pmatrix}\begin{pmatrix}2 & 2 & 1\\ 3 & -3 & 0\\ 7 & 2 & -1\\ -4 & -5 & 0\end{pmatrix} = \begin{pmatrix}7 & 1 & 2\\ 19 & 2 & 2\\ -23 & -36 & 1\end{pmatrix}.$$

$$d.\quad \begin{pmatrix}2 & 1 & 1\end{pmatrix}\begin{pmatrix}3 & 1 & -1\\ 2 & 1 & 1\\ 1 & 2 & 2\end{pmatrix}\begin{pmatrix}5\\ -1\\ -1\end{pmatrix} = 39.$$

3.2.1.1. *Laws of matrix algebra.*

The basic arithmetic of matrix algebra is not so different than the algebra of real numbers. Let A, B, and C be three matrices such that the following operations are well defined.

(1) $(A+B)+C = A+(B+C)$.
(2) $(AB)\,C = A\,(BC)$.
(3) $A+B = B+A$.
(4) $A\,(B+C) = AB+AC$.

However, we should point out that some operations are not similar to commonly-used algebra. For example, for any two numbers r_1, r_2, we know that $r_1 \cdot r_2 = r_2 \cdot r_1$. But the same does not hold when matrices are involved. In fact, the operations might not even be well defined when reversing the order of matrices.

3.2.1.2. *The representation of basic operations by matrices.*

The three elementary operations can be represented by matrices, such that each basic operation on a matrix A, it equivalent to left multiplication with the relevant matrix. For example. if one wants to interchange the i^{th} and j^{th} row of a matrix A, one can left multiply A by the matrix B with entries

$$b_{kl} = \begin{cases} 1, & k = l \neq i, j,\\ 1, & k = i, l = j,\\ 1 & k = j, l = i\\ 0, & \text{otherwise.}\end{cases}$$

This matrix B is a matrix where almost all entries are 0, except for the diagonal where all entries are 1 (excluding b_{ii} and b_{jj}), and $b_{ij} = b_{ji} = 1$. If one wants to multiply the i^{th} row with a number r, one can left multiply A by a matrix B with entries

$$b_{kl} = \begin{cases} 1, & k = l \neq i, \\ r, & k = l = i, \\ 0, & \text{otherwise.} \end{cases}$$

And one can also use left multiplication to add r times row i to row j by using the matrix B where

$$b_{kl} = \begin{cases} 1, & k = l, \\ r, & k = j, l = i, \\ 0, & \text{otherwise.} \end{cases}$$

For example, consider the generic 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Say we want to multiply the second raw by 5. We can use the following left multiplication

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 5a_{21} & 5a_{22} & 5a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

as required. Now assume we want to add 5 times the second row to the third row. Then

$$BA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 5a_{21} + a_{31} & 5a_{22} + a_{32} & 5a_{23} + a_{33} \end{pmatrix}.$$

3.2.2. Special matrices.

- *The Identity matrix* I is a square matrix, which means that then umber of rows equals then umber of columns, such that

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

  All the non-diagonal entries are 0, and all the entries along the diagonal are 1. The most basic and important property of this matrix is, that for every matrix A, it follows that $AI = A$ (assuming that the multiplication is well defined). We will usually denoted the $n \times n$ identity matrix by I_n to state its dimensions. For any $n \times n$ matrix A, we get $AI = IA$.
- The identity matrix is a special form of *the Diagonal matrix* where all non-diagonal entries are 0,

$$D = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}.$$

The interesting property of diagonal matrices is their form when multiplying by themselves. For example,

$$D^k = D \cdot D \cdots D = \begin{pmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn}^k \end{pmatrix}.$$

This property will prove useful later on.

- In addition, we have the *Upper-Triangular matrix* (and, *Lower-Triangular matrix*) where every entry $a_{ij} = 0$ if $i > j$ (if $i < j$). That is, all entries above (below) the diagonal are zeros. Note that there product remains a triangular matrix.

$$\begin{aligned} \text{An Upper}-\text{Triangular matrix} &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \\ \text{for example,} & \begin{pmatrix} 0 & 2 & 17 & 3 \\ 0 & -1 & 9 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{A Lower}-\text{Triangular matrix} &= \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \\ \text{for example,} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 8 & 3 & 5 & 0 \\ 0 & 4 & 7 & -6 \end{pmatrix}. \end{aligned}$$

EXERCISE 3.15. Let

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(1) Compute $A + B,\ A - D,\ 3B,\ CE,\ EC,\ -D$.
(2) Verify that $CD \neq DC$.

Solution.

(1) A direct computation shows that

$$\begin{aligned}
A+B &= \begin{pmatrix} 2 & 4 & 0 \\ 4 & -2 & 4 \end{pmatrix}. \\
A-D &= \text{undefined.} \\
3B &= \begin{pmatrix} 0 & 3 & -3 \\ 12 & -3 & 6 \end{pmatrix} \\
CE &= \begin{pmatrix} -1 \\ 4 \end{pmatrix} \\
EC &= \text{undefined.} \\
-D &= \begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}
\end{aligned}$$

(2) We can see that

$$CD = \begin{pmatrix} 4 & 3 \\ 5 & 2 \end{pmatrix} \neq \begin{pmatrix} 5 & 3 \\ 4 & 1 \end{pmatrix} = DC.$$

EXERCISE 3.16. Show that if B is a scalar multiple of the 2×2 identity matrix, then $AB = BA$ for all 2×2 matrices A.

Solution. Fix $B = rI$ when r is a real number. Using the laws of matrix algebra yields

$$BA = (rI)\,A = r\,(IA) = rA = Ar = AIr = AB.$$

EXERCISE 3.17.

(1) Prove that $(AB)^k = A^k B^k$ if $AB = BA$.
(2) Show that $(AB)^k \neq A^k B^k$ in general.
(3) Conclude that $(A+B)^2$does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

Solution.

(1) Proof by induction. Clearly the statement hold for $k = 1$. assume it holds for k. That is, $(AB)^{k-1} = A^{k-1}B^{k-1}$. Applying $AB = BA$ and using the induction hypothesis, we easily find

$$\begin{aligned}
(AB)^k &= (AB)^{k-1}\,(AB) \\
&= A^{k-1}B^{k-1}\,(BA) \\
&= A^{k-1}B^{k-1}BA \\
&= A^{k-1}B^kA.
\end{aligned}$$

Now we can apply $AB = BA$ for k times on $B^kA = AB^k$ and get

$$(AB)^k = A^{k-1}B^kA = A^{k-1}AB^k = A^kB^k.$$

(2) Take, for example,

$$A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix},\ B = A = \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}.$$

For $k = 2$, we get

$$\begin{aligned}
AB &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}, \\
(AB)^2 &= \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 5 \\ 30 & 10 \end{pmatrix},
\end{aligned}$$

while

$$\begin{aligned} A^2 &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \\ B^2 &= \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}, \\ A^2B^2 &= \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 6 & 6 \end{pmatrix} \neq \begin{pmatrix} 15 & 5 \\ 30 & 10 \end{pmatrix} = (AB)^2 . \end{aligned}$$

(3) By the previous answers we can say that

$$\begin{aligned} (A+B)^2 &= (A+B)(A+B) \\ &= A^2 + AB + BA + B^2. \end{aligned}$$

And unless $AB = BA$, then $(A+B)^2 \neq A^2 + 2AB + B^2$.

3.3. Transpose and invariability

3.3.1. The transpose matrix.

Let A be a $m \times n$ matrix, such that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

The transpose of A, denoted A^T, is a $n \times m$ matrix such that

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}.$$

In words, we reversing the order of values of A, such that the rows became the columns of A^T and vice versa.

There are a few basic rules for the transpose matrix. that one could easily prove by a direct computation. Assume that A and B are two matrices such that the following operations are well defined.

(1) $(A \pm B)^T = A^T \pm B^T$.
(2) $\left(A^T\right)^T = A$.
(3) $\left(rA^T\right) = rA^T$ for every real number r.
(4) $(AB)^T = B^T A^T$.

EXERCISE 3.18. Let

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \text{and} \qquad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(1) Compute B^T, A^TC^T, $(CA)^T$, $(CE)^T$, E^TC^T.
(2) Verify that $(DA)^T = A^TD^T$.

Solution.

(1) A direct computation and usage of the definitions shows:

$$\begin{aligned} B^T &= \begin{pmatrix} 0 & 4 \\ 1 & -1 \\ -1 & 2 \end{pmatrix}, \\ (CA)^T = A^T C^T &= \begin{pmatrix} 2 & 6 \\ 1 & 10 \\ 5 & 1 \end{pmatrix}. \\ E^T C^T = (CE)^T &= \begin{pmatrix} -1 & 4 \end{pmatrix}. \end{aligned}$$

(2) First compute DA.

$$DA = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 4 \\ 2 & 2 & 3 \end{pmatrix}.$$

Now, see that

$$A^T D^T = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 4 & 3 \end{pmatrix} = (DA)^T.$$

The transpose operations enables us to define a few more special matrices. A matrix A is called *symmetric* if $A^T = A$, and it is called *anti-symmetric* if $A^T = -A$.

3.3.2. Invertible matrices.

Denoted the class of $n \times n$ matrices by M_n.

DEFINITION 3.2. **(Inverse matrix)** Fix $A \in M_n$. If there exists a matrix $B \in M_n$ such that $AB = BA = I$, then B is the *inverse* of A and both matrices are invertible.

We denote that inverse matrix of A by A^{-1}. In general, a matrix $A \in M_n$ could have a *right inverse matrix* B where $AB = I$. It could also have a *left inverse matrix* B where $BA = I$. If the matrix A has a left inverse and a right inverse then it is invertible.

LEMMA 3.1. *A matrix $A \in M_n$ can have at most one inverse matrix.*

There are several properties of inverse matrices presented in the following claim.

CLAIM 3.1. Let $A, B \in M_n$ be two invertible matrices. Then

- $\left(A^{-1}\right)^{-1} = A$.
- $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$.
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
- For every real number $r \neq 0$, it follows that $(rA)^{-1} = \frac{1}{r}A^{-1}$.

Invertible matrices are very useful when trying to find a solution to a system of linear equations. The following theorem explains this.

EXERCISE 3.19. Prove that

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}.$$

Solution. We can verify this easily by the definition of an inverse matrix.

$$\begin{aligned} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

EXERCISE 3.20. Check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{pmatrix}.$$

Solution. Using the definition of an inverse matrix we see that

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 0.5 & 0 & -0.5 \\ 0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.5 & 0 & -0.5 \\ 0.5 & 0 & 0.5 \\ -0.5 & 1 & -0.5 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

3.3.3. Finding the inverse matrix.

Although the inverse matrices are quite useful, finding them is not always an easy task. for that case, we have the Gauss-Jordan elimination process. An identity matrix I is placed along side a matrix A that is to be inverted. Then, the same elementary row operations are performed on both matrices until A has been reduced to an identity matrix. The identity matrix upon which the elementary row operations have been performed will then become the inverse matrix we seek.

EXAMPLE 3.3. Assume we wish to inverse the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

First, we write this matrix augmented with the identity matrix,

$$[A|I] = \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix}.$$

Then we preform the row operations on this matrix to reduce A to its row echelon form.

$$\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 3 & 4 & | & 0 & 1 \end{pmatrix} \xrightarrow{L_2 - 3L_1 \to L_2} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -2 & | & -3 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & -2 & | & -3 & 1 \end{pmatrix} \xrightarrow{-\frac{1}{2}L_2 \to L_2} \begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \xrightarrow{L_1 - 2L_2 \to L_1} \begin{pmatrix} 1 & 0 & | & -2 & 1 \\ 0 & 1 & | & -\frac{3}{2} & \frac{1}{2} \end{pmatrix}.$$

On the RHS we got the inverse of A as we already shown.

EXERCISE 3.21. Invert the following matrices:

$$\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} -4 & 1 \\ 2 & -4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

Solution. Use the method we sew, one gets

$$\begin{pmatrix} 2 & 3 \\ -2 & 1 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 1 & -3 \\ 2 & 2 \end{pmatrix}.$$

$$\begin{pmatrix} -4 & 1 \\ 2 & -4 \end{pmatrix}^{-1} = \frac{1}{14}\begin{pmatrix} -4 & -1 \\ -2 & -4 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & 1 \\ 1 & 0 & -1 \end{pmatrix}.$$

$$\begin{pmatrix} 5 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}^{-1} = \frac{1}{25}\begin{pmatrix} 5 & -4 & 3 \\ 0 & 15 & -5 \\ 0 & -5 & 10 \end{pmatrix}.$$

EXERCISE 3.22. Assume that $ad - bc \neq 0$ and fix a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Find A^{-1}.

Solution. Note that if $a = c = 0$, then the matrix is singular and therefore not invertible. Thus, we can assume that $a \neq 0$ (otherwise we can just exchange the rows). If we divide the first row by a and add $-c$ times the first row to the second row, we get

$$\begin{pmatrix} a & b & | & 1 & 0 \\ c & d & | & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & \frac{b}{a} & | & \frac{1}{a} & 0 \\ 0 & \frac{da-cb}{a} & | & -\frac{c}{a} & 1 \end{pmatrix}.$$

Multiply the second row by $\frac{a}{ad-bc}$ to get

$$\begin{aligned}\begin{pmatrix} a & \frac{b}{a} & | & \frac{1}{a} & 0 \\ 0 & 1 & | & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} &\xrightarrow{L_1 - \frac{b}{a}L_2 \to L_1} \begin{pmatrix} a & 0 & | & \frac{1}{a}\left(1 + \frac{cb}{ad-bc}\right) & \frac{-b}{ad-bc} \\ 0 & 1 & | & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} \\ &= \begin{pmatrix} a & 0 & | & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & | & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}.\end{aligned}$$

Thus,

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

EXERCISE 3.23. What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}?$$

Solution. The inverse is

$$D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{pmatrix}.$$

Clearly we need to assume that $d_i \neq 0$ otherwise the rank is smaller than n, and the matrix is not invertible.

3.4. The rank of a matrix

The concept of a rank of a matrix is important when considering the number of solutions a system might have.

DEFINITION 3.3. **(Rank)** The *rank* of a matrix is the number of nonzero rows in its row echelon form.

A few simple properties of rank:

(1) let A be the coefficient matrix and $\hat{A}$ be the corresponding augmented matrix. Then,

$$\operatorname{rank}(A) \quad \leq \quad \min\left\{\operatorname{rank}\hat{A}, \#\left\{\text{columns in }\hat{A}\right\}, \#\left\{\text{rows in }\hat{A}\right\}\right\}.$$

(2) For any two matrices A, B such that AB is well defined, it follows that

$$\operatorname{rank}(AB) \leq \min\left(\operatorname{rank}(A), \operatorname{rank}(B)\right),$$

(3) $\operatorname{rank}(A) = \operatorname{rank}\left(A^T\right) = \operatorname{rank}\left(AA^T\right) = \operatorname{rank}\left(A^TA\right).$

EXERCISE 3.24. Compute the rank of the following matrices:

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix}, \quad \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix}$$

Solution. We find the row echelon form of each matrix and derive the rank from it.

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 2 & -4 \\ 0 & 0 \end{pmatrix} \Rightarrow \text{so its rank is 1.}$$

$$\begin{pmatrix} 2 & -4 & 2 \\ -1 & 2 & 1 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 2 & -4 & 2 \\ 0 & 0 & 2 \end{pmatrix} \Rightarrow \text{so its rank is 2.}$$

$$\begin{pmatrix} 1 & 6 & -7 & 3 \\ 1 & 9 & -6 & 4 \\ 1 & 3 & -8 & 4 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 6 & -7 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \Rightarrow \text{so its rank is 3.}$$

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 1 & 9 & -6 & 4 & 9 \\ 1 & 3 & -8 & 4 & 2 \\ 2 & 15 & -13 & 11 & 16 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 6 & -7 & 3 & 5 \\ 0 & 3 & 1 & 1 & 4 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{so its rank is 3.}$$

$$\begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 1 & 9 & -6 & 4 & 2 \\ 1 & 3 & -8 & 4 & 5 \end{pmatrix} \quad \rightarrow \quad \begin{pmatrix} 1 & 6 & -7 & 3 & 1 \\ 0 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} \Rightarrow \text{so its rank is 3.}$$

EXERCISE 3.25. Find the rank of the following matrices:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 4 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \\ -1 & 0 & 2 & 1 \end{pmatrix}.$$

Solution. The rank of A is 2, and the rank of B is 3.
Now we can use the concept of rank to determine then number of solutions to a system. In case a solution exists, the next question that arises is whether it is unique. For that matter we define the concept of *non-singularity.*

DEFINITION 3.4. **(Non-singular matrix)** A coefficient matrix A is called *non-singular* if for every choice of the RHS $b_1, b_2, \ldots, b_m$ its corresponding system $A\mathbf{x} = \mathbf{b}$ of linear equations has exactly one solution.

The following theorem characterizes systems of linear equations that have a solution using the concept of rank.

THEOREM 3.1. *A system of linear equations with coefficient matrix A and augmented matrix $\hat{A}$ has a solution if and only if* $\operatorname{rank}(\hat{A}) = \operatorname{rank}(A)$. *Moreover, a coefficient matrix A is non-singular if and only if*

$$\text{number of rows of } A = \text{number of columns of } A = \operatorname{rank}(A).$$

An important type of systems in this context are homogeneous ones where $b_i = 0$ for every i.

DEFINITION 3.5. **(Homogeneous system)** If $b_i = 0$ for every i, then system $A\mathbf{x} = \mathbf{b}$ is *homogeneous.*

One can verify that every homogeneous system always has at least one solution, which is the trivial solution where $x_i = 0$ for every unknown i. On the other hand, a non-homogeneous system may not have a solution. For example, in Exercise 3.12 we sew that for every $k \neq -8$, the system

$$\begin{cases} 6x + y & = 7 \\ 3x + y & = 4 \\ -6x - 2y & = k \end{cases}$$

did not have a solution, as its reduced echelon form is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 8+k \end{pmatrix}.$$

THEOREM 3.2. *For every $A \in M_n$, the following statements are equivalent:*

(1) A is invertible.
(2) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is $\mathbf{x} = \mathbf{0}$.
(3) For every vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.
(4) A is non-singular.
(5) $\operatorname{rank}(A) = n$.

EXERCISE 3.26. Fix a triangular matrix A. Prove that A is invertible if and only if all its diagonal entries are non zero.

SOLUTION. Note that we can restrict attention to upper triangular matrices, since rank and diagonal entries do not change when taking the transpose of a matrix. Assume that all diagonal entries are non zero. Thus, the row-echelon form of A is with n non-zero rows. Thus implies that $\operatorname{rank}(A) = n$, hence the matrix in invertible. On the other hand, if there exists a zero diagonal entry, then we can perform the G-J elimination process and eliminate at least one row (if not other rows, than the row with the zero diagonal entry). Thus, $\operatorname{rank}(A) < n$, and the matrix is in invertible.

EXERCISE 3.27.

(1) Show that the inverse of a 2×2 lower-triangular matrix is a lower-triangular matrix.
(2) Show that the inverse of a 2×2 upper-triangular matrix is a upper-triangular matrix.

Solution.

(1) Fix a lower-triangular matrix $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$. Denote $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw & ax \\ bw + cy & bx + cz \end{pmatrix}.$$

We require that

$$\begin{aligned} aw &= 1, \\ ax &= 0, \\ bw + cy &= 0, \\ bx + cz &= 1. \end{aligned}$$

Clearly, $x = 0$ as $a \neq 0$. Since the inverse is unique and exists, the result follows.

(2) Fix an upper-triangular matrix $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. Denote $A^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + cz \\ cy & cz \end{pmatrix}.$$

We require that

$$\begin{aligned} aw + by &= 1, \\ ax + cz &= 0, \\ cy &= 0, \\ cz &= 1. \end{aligned}$$

Clearly, $y = 0$ as $c \neq 0$. Since the inverse is unique and exists, the result follows.

3.4.1. Number of solutions - a short review. Since there are numerous theorems and corollaries about the number of solutions of systems of linear equations, we give the short survey in the following table. Let m be the number of equations in $A\mathbf{x} = \mathbf{b}$ (this is basically the number of rows of A), let n be the number of unknowns $x_1, x_2, \ldots, x_n$. The following table presents the number of solutions a system has, as a function of the relations between m and n and the conditions given in the LHS column.

	$m < n$	$m = n$	$m > n$
If $b_i = 0$ for every i,	∞	$1, \infty$	$1, \infty$
For every $\mathbf{b}$,	$0, \infty$	$0, 1, \infty$	$0, 1, \infty$
For every $\mathbf{b}$, if $\operatorname{rank}(A) = m$,	∞	1	***
For every $\mathbf{b}$, if $\operatorname{rank}(A) = n$,	***	1	$0, 1$

TABLE 1. Number of solutions in different systems of linear equations.

EXERCISE 3.28. The following five matrices are coefficient matrices of systems of linear equations. For each matrix, what can you say about the number of solutions of the corresponding system when:

(1) The system is homogeneous.
(2) The system is not homogeneous.

$$A = \begin{pmatrix} 2 & -4 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 1 \\ 1 & 4 \\ 0 & 3 \end{pmatrix},$$

$$D = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 1 & 0 \\ 0 & 7 & 6 \end{pmatrix}.$$

Solution. We start with the case where the system is homogeneous.

(1) rank (A) = #rows = #columns, thus there is a unique solution $(0, 0)$.
(2) rank (B) = #rows < #columns, thus there are infinitely many solutions.
(3) rank (C) = #columns, thus there is a unique solution $(0, 0)$.

(4) rank (D) = #rows = #columns, thus there is a unique solution $(0, 0)$.

(5) rank (E) < #rows = #columns, thus there are infinitely many solutions.

In case the system is not homogeneous:

(1) rank (A) = #rows = #columns, thus there is a unique solution.

(2) rank (B) = #rows < #columns, thus there are infinitely many solutions.

(3) rank (C) = #columns, thus there either zero solutions or one solution.

(4) rank (D) = #rows = #columns, thus there is a unique solution $(0, 0)$.

(5) rank (E) < #rows = #columns, thus either zero solutions or infinitely many solution..

CHAPTER 4

Determinants

4.1. Defining the determinant

The determinant of a matrix $A \in M_n$ is a value attached to each square matrix. The computation of this value tends to be complex but it has great significance. The determinant is a value that enables us to know whether a square matrix is non-singular. More specifically, if the determinant of a matrix A is not 0, then we know that it is invertible, non-singular, and thus the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

The determinant is defined inductively. We first define the determinant for a 1×1 matrix, and then we use this definition to find the determinant of a 2×2 matrix and so on. In order to do so, for every matrix $A \in M_n$, let A_{ij} be the $(n-1) \times (n-1)$ sub-matrix obtained from A by deleting its i^{th} row and j^{th} column. For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix},$$

then

$$A_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix},\ A_{23} = \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix},\ A_{32} = \begin{pmatrix} 1 & 3 \\ 4 & 6 \end{pmatrix},\ \text{and } A_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}.$$

In case of a 1×1 matrix , the determinant of $A \in M_1$ (which is basically a number, $A = (a)$) is $\det(A) = a$. The determinant of any $A \in M_n$ is

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} \cdot a_{ij} \cdot \det(A_{ij})$$

for any $1 \leq i \leq n$. That is, we can choose $i = 1, 2, \ldots, n$ and compute the determinants of all $A_{ij}s$ and use these values to compute $\det(A)$.

For example, fix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then

$$\begin{aligned} \det(A) &= \sum_{j=1}^{2} (-1)^{2+j} \cdot a_{2j} \cdot \det(A_{2j}) \\ &= (-1)^{2+1} \cdot a_{21} \cdot \det(A_{21}) + (-1)^{2+2} \cdot a_{22} \cdot \det(A_{22}) \\ &= -a_{21}a_{12} + a_{22}a_{11}. \end{aligned}$$

One can see that even if we were to choose $i = 1$, the result remains the same as

$$\begin{aligned} \det(A) &= \sum_{j=1}^{2} (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}) \\ &= (-1)^{1+1} \cdot a_{11} \cdot \det(A_{11}) + (-1)^{1+2} \cdot a_{12} \cdot \det(A_{12}) \\ &= a_{11}a_{22} - a_{12}a_{21}. \end{aligned}$$

Therefore, the determinant is well defined. In case $A \in M_3$, we get

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{3} (-1)^{1+j} \cdot a_{1j} \cdot \det(A_{1j}) \\
&= a_{11}\det(A_{11}) + a_{12}\det(A_{12}) + a_{13}\det(A_{13}) = \\
&= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{12}(a_{21}a_{32} - a_{22}a_{31}),
\end{aligned}$$

and we used the previous result for 2×2 matrices.

4.2. Properties of determinants

Determinant are not easily computable, therefore we have a set of properties that assist with this computation. Let $A, B \in M_n$ be two square matrices.

- $\det(AB) = \det(A)\det(B)$.
- $\det(A^T) = \det(A)$.
- $\det(A+B) \neq \det(A) + \det(B)$, in general.
- $\det(A) = \pm\det(R)$, when R is the row echelon form of A. If no row interchange was used, then $\det(A) = \det(R)$.
- $\det(A) = k \cdot \det(R)$, when R is reached by taking A and multiplying one row by the real number k.
- $\det(A) = \pm\det(R)$, when R is reached by interchanging any two rows (columns) of A.
- $\det(A) = \det(R)$, when R is reached by adding a multiple of any row to another row.
- The determinant of any diagonal matrix, lower-triangular matrix, or upper-triangular matrix is the product of its diagonal entries.
- $\det(A) = 0$ if A has two identical rows.

Exercise 4.1. Find the determinants of the matrices:

$$\begin{pmatrix} 2 & 1 \\ -4 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ -4 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 1 & 0 \\ -2 & 7 & -2 \\ 2 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 0 & 6 & -1 \end{pmatrix}.$$

Solution. The determinants are:

$$\begin{aligned}
\det\begin{pmatrix} 2 & 1 \\ -4 & 5 \end{pmatrix} &= 14. \\
\det\begin{pmatrix} 3 & 6 \\ -4 & -1 \end{pmatrix} &= 21. \\
\det\begin{pmatrix} 3 & 1 & 0 \\ -2 & 7 & -2 \\ 2 & 0 & 6 \end{pmatrix} &= 134. \\
\det\begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 0 & 6 & -1 \end{pmatrix} &= -12.
\end{aligned}$$

4.2.1. Determinants: applications.

The following theorem, combing with Theorem 3.2, shows how the determinant assists with solving systems of linear equations and finding the inverse of a matrix.

4.2.1.1. *Computing the inverse.*

Theorem 4.1. *For every $A \in M_n$. The matrix A is invertible if and only if* $\det(A) \neq 0$.

Thus, we can use the equivalences given in Theorem 3.2 also when $\det(A) \neq 0$. Moreover, the determinant is not only useful to determine whether a matrix A is invertible, it is also helpful with finding A^{-1}.

DEFINITION 4.1. **(Co-factor and adjoint matrix)** Fix a matrix $A \in M_n$. The *co-factor* $C_{ij} = (-1)^{i+j}\det(A_{ij})$ is the determinant of the matrix A after deleting the i^{th} row and j^{th} column. The *adjoint matrix* $\operatorname{adj}(A)$ of A is

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T .$$

That is, the $(i,j)^{\text{th}}$ entry of $\operatorname{adj}(A)$ is C_{ji} (note that the indices are switched by the transpose).

As the following theorem states, the adjoint matrix is the inverse of the original matrix, up to a factor.

THEOREM 4.2. *If A is an invertible matrix, then $A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$.*

4.2.1.2. *Cramer's rule for solving a system of linear equations.* The following theorem is known by the name: *Cramer's rule.*

THEOREM 4.3. **(Cramer's Rule)** *Fix an invertible matrix $A \in M_n$. The unique solution $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ to the system $A\mathbf{x} = \mathbf{b}$ is*

$$x_i = \frac{\det(B_i)}{\det(A)},$$

for every $i = 1, 2, \ldots, n$, where B_i is the matrix A with the vector $\mathbf{b}$ replacing the i^{th} column of A.

EXERCISE 4.2. Use Cramer's rule to solve the following systems of equations:

(1)

$$\begin{aligned} 6x - 2y - 3z &= 1 \\ 2x + 4y + z &= -2 \\ 3x - z &= 8. \end{aligned}$$

(2)

$$\begin{aligned} 5x - 2y + z &= 9 \\ 3x - y &= 9 \\ 3y + 2z &= 15. \end{aligned}$$

Solution.

(1) Using Cramer's rule we get

$$\begin{pmatrix} 6 & -2 & -3 \\ 2 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 8 \end{pmatrix},$$

and

$$x = \frac{\det \begin{pmatrix} 1 & -2 & -3 \\ -2 & 4 & 1 \\ 8 & 0 & -1 \end{pmatrix}}{\det \begin{pmatrix} 6 & -2 & -3 \\ 2 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix}} = 40,$$

$$y = \frac{\det\begin{pmatrix} 6 & 1 & -3 \\ 2 & -2 & 1 \\ 3 & 8 & -1 \end{pmatrix}}{\det\begin{pmatrix} 6 & -2 & -3 \\ 2 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix}} = \frac{-97}{2},$$

$$z = \frac{\det\begin{pmatrix} 6 & -2 & 1 \\ 2 & 4 & -2 \\ 3 & 0 & 8 \end{pmatrix}}{\det\begin{pmatrix} 6 & -2 & -3 \\ 2 & 4 & 1 \\ 3 & 0 & -1 \end{pmatrix}} = 112.$$

(2) Using Cramer's rule we get

$$\begin{pmatrix} 5 & -2 & 1 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \\ 5 \end{pmatrix},$$

and

$$x = \frac{\det\begin{pmatrix} 9 & -2 & 1 \\ 9 & -1 & 0 \\ 5 & 3 & 2 \end{pmatrix}}{\det\begin{pmatrix} 5 & -2 & 1 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} = \frac{60}{11},$$

$$y = \frac{\det\begin{pmatrix} 5 & 9 & 1 \\ 3 & 9 & 0 \\ 0 & 5 & 2 \end{pmatrix}}{\det\begin{pmatrix} 5 & -2 & 1 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} = \frac{81}{11},$$

$$z = \frac{\det\begin{pmatrix} 5 & -2 & 9 \\ 3 & -1 & 9 \\ 0 & 3 & 5 \end{pmatrix}}{\det\begin{pmatrix} 5 & -2 & 1 \\ 3 & -1 & 0 \\ 0 & 3 & 2 \end{pmatrix}} = -\frac{39}{11}.$$

EXERCISE 4.3. For each of the following matrices, compute the row echelon form and verify that $\det(A) = \pm\det(R)$, when R is the row echelon form of A. Remember that if no row interchange was used, then $\det(A) = \det(R)$.

$$\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 0 & 7 & 8 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 1 & 9 & 6 \end{pmatrix}.$$

Solution.

$$\det\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix} = \det\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix} = -1.$$

$$\det\begin{pmatrix}2 & 4 & 0\\ 4 & 6 & 3\\ -6 & -10 & 0\end{pmatrix} = \det\begin{pmatrix}2 & 4 & 0\\ 0 & -8 & 3\\ 0 & 0 & 3/4\end{pmatrix} = -12.$$

$$\det\begin{pmatrix}0 & 1 & 2\\ 3 & 4 & 5\\ 0 & 7 & 8\end{pmatrix} = -\det\begin{pmatrix}3 & 4 & 5\\ 0 & 1 & 2\\ 0 & 0 & -6\end{pmatrix} = 18.$$

$$\det\begin{pmatrix}1 & 1 & 1\\ 1 & 4 & 2\\ 1 & 4 & 3\end{pmatrix} = \det\begin{pmatrix}1 & 1 & 1\\ 0 & 3 & 1\\ 0 & 0 & 1\end{pmatrix} = 3.$$

$$\det\begin{pmatrix}1 & 1 & 1\\ 0 & 4 & 5\\ 1 & 9 & 6\end{pmatrix} = \det\begin{pmatrix}1 & 1 & 1\\ 0 & 4 & 5\\ 0 & 0 & -5\end{pmatrix} = -20.$$

4.3. Linear independence

Let $v_1, v_2, \ldots, v_n$ be n vectors in $\mathbb{R}^n$. A convex combination of $v_1, \ldots, v_n$ is the sum $\sum_{i=1}^n \alpha_i v_i$ when $\alpha_i \in \mathbb{R}$ for every $i = 1, \ldots, n$. We say that the vectors $v_1, \ldots, v_n$ are *linearly dependent* if there exists n numbers $\alpha_1, \ldots, \alpha_n$ (not all of them are zeros) such that $\sum_{i=1}^n \alpha_i v_i = 0$. If such n do not exist, then the vectors are *linearly independent*.

Linear independence of vectors has a strong connection to the rank of a matrix, and thus to its determinant, and the number of solution a system of linear equations might have. In Definition 3.3 we defined the rank as the number of nonzero rows in its row echelon form. The following lemma gives present some equivalences for this value.

LEMMA 4.1. *The rank of a matrix A, denoted* $\operatorname{rank}(A)$ *, equals the maximal number of linearly independent rows (and also, columns).*

THEOREM 4.4. *If $k > n$, any set of k vectors in $\mathbb{R}^n$ are linearly dependent.*

In Theorem 3.2 we presented a few equivalent properties of a matrix. In Theorem 4.1 we added another equivalent property. The following theorem presents all the equivalent properties, including independence.

THEOREM 4.5. *For every $A \in M_n$, the following statements are equivalent:*

(1) A is invertible.
(2) $\det(A) \neq 0$
(3) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, which is $\mathbf{x} = \mathbf{0}$.
(4) For every vector $\mathbf{b}$, the system $A\mathbf{x} = \mathbf{b}$ has exactly one solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.
(5) A is non-singular.
(6) $\operatorname{rank}(A) = n$.
(7) The n vectors that are the n columns of A are linearly independent.

In case, one of the previous statements does not hold, and specifically in case that $\det(A) = 0$, then we use Theorem 3.1. The theorem states that for every system of linear equations with coefficient matrix A and augmented matrix $\hat{A}$, a solution exits if and only if $\operatorname{rank}(\hat{A}) = \operatorname{rank}(A)$.

EXERCISE 4.4. Which of the following pairs or triplets of vectors are linearly independent?

(1)
$$(2,1), (1,2).$$

(2)
$$(2,1),(-4,-2).$$
(3)
$$(1,1,0),(0,1,1).$$
(4)
$$(1,1,0),(0,1,1),(1,0,1).$$

Solution.

(1) We can write down the equations
$$c_1(2,1)+c_2(1,2)=(0,0).$$
In the matrix form, we get
$$\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$
The only solution is $(0,0)$, and the vectors are linearly independent.

(2) Writing down the equations in the matrix form
$$\begin{pmatrix}2 & -4\\ 1 & -2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}.$$
There is a solution $(-2,1)$, and the vectors are linearly dependent.

(3) The equations are
$$\begin{pmatrix}1 & 0\\ 1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix},$$
and the only solution is $(0,0)$. Linearly independent.

(4) The equations are
$$\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\\ 0 & 1 & 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}0\\ 0\\ 0\end{pmatrix}.$$
Again, the trivial solution is the only solution, and the vectors are linearly independent.

EXERCISE 4.5. Which of the following triplets of vectors are linearly independent?

(1)
$$(1,0,1,0),(1,0,0,1),(0,0,1,1).$$
(2)
$$(1,0,1,0),(1,0,-1,0),(1,0,0,0).$$

Solution.

(1) The coefficient matrix is
$$\begin{pmatrix}1 & 1 & 0\\ 0 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 1\end{pmatrix}.$$
The rank of this matrix is 3, so the homogeneous system has only the trivial solution. Thus, the vectors are independent.

(2) The coefficient matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The rank of this matrix is 2, so the homogeneous system has infinitely many solutions. Thus, the vectors are dependent.

EXERCISE 4.6. Prove that any collection of vectors that includes the zero-vector cannot be linearly independent.

Solution. Let $\{v_1, \ldots, v_k\}$ be a collection of vector where v_1 is the zero-vector. Then

$$1 \cdot v_1 + 0 \cdot v_2 + \cdots + 0 \cdot v_k = (0, 0, \ldots, 0),$$

and the vectors are linearly dependent.

4.4. General exercises

EXERCISE 4.7. Use the adjoint matrix to find A^{-1} of

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $\det(A) \neq 0$.

Solution. We know that $\det(A) = ad - bc \neq 0$.

$$\begin{aligned} \operatorname{adj}(A)_{11} = d \quad &, \quad \operatorname{adj}(A)_{22} = a, \\ \operatorname{adj}(A)_{12} = -b, \quad &, \quad \operatorname{adj}(A)_{21} = -c. \end{aligned}$$

Thus,

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\ &= \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \end{aligned}$$

EXERCISE 4.8. Determine the number of solution the following systems have:

(1)
$$\begin{pmatrix} 3 & 6 & 0 \\ 2 & 0 & -5 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ -10 \end{pmatrix}.$$

(2)
$$\begin{pmatrix} 4 & -1 & 8 \\ 17 & -8 & 10 \\ -3 & 2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 160 \\ 200 \\ 40 \end{pmatrix}.$$

(3)
$$\begin{pmatrix} 2 & -3 & 0 \\ 3 & 0 & 5 \\ 2 & 6 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 15 \\ 18 \end{pmatrix}.$$

(4)
$$\begin{pmatrix} 4 & -1 & 8 \\ 3 & 0 & 2 \\ 5 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}.$$

(5)
$$\begin{pmatrix} 6 & -1 & -1 \\ 5 & 2 & -2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ 4 \end{pmatrix}.$$

Solution.

(1) The determinant is
$$\det \begin{pmatrix} 3 & 6 & 0 \\ 2 & 0 & -5 \\ 1 & -1 & -1 \end{pmatrix} = -33,$$
thus the solution is unique.

(2) The determinant of the matrix is 0 so we need to find whether there are an infinite number of solutions, or no solution at all. The row echelon form of the augmented matrix is
$$\begin{pmatrix} 4 & -1 & 8 & | & 160 \\ 0 & -15/4 & -24 & | & -480 \\ 0 & 0 & 0 & | & 0 \end{pmatrix},$$
so there are infinitely many solutions.

(3) The determinant of the matrix is 0, and the row echelon form
$$\begin{pmatrix} 2 & -3 & 0 & | & 6 \\ 3 & 0 & 5 & | & 15 \\ 0 & 0 & 0 & | & 0 \end{pmatrix},$$
suggest that there are infinitely many solutions.

(4) The determinant of the matrix is 0. The row echelon form of the augmented matrix is
$$\begin{pmatrix} 4 & -1 & 8 & | & 30 \\ 3 & 0 & 2 & | & 20 \\ 0 & 0 & 0 & | & 10 \end{pmatrix},$$
suggest that there are no solutions.

(5) The determinant of the matrix is -27, so there is a unique solution.

Exercise 4.9. Find the values of a for which the following matrices do not have an inverse.

(1)
$$\begin{pmatrix} 6 & -1 \\ 2 & a \end{pmatrix}.$$

(2)
$$\begin{pmatrix} 5 & a & 0 \\ 4 & 2 & 1 \\ -1 & 3 & 1 \end{pmatrix}.$$

(3)
$$\begin{pmatrix} 5 & 3 \\ -3 & a \end{pmatrix}.$$

(4)
$$\begin{pmatrix} -1 & 3 & 1 \\ 0 & 5 & a \\ 6 & 2 & 1 \end{pmatrix}.$$

Solution.

(1) There is no inverse if the determinant is 0, which leads to the equation
$$6a + 2 = 0.$$

Thus, $a = -\frac{1}{3}$.

(2) Setting the determinant equal to zero and solving for a yields $a = -1$.

(3) There is no inverse if the determinant is 0, which leads to the equation

$$5a + 9 = 0.$$

Thus, $a = -\frac{9}{5}$.

(4) Setting the determinant equal to zero and solving for a yields $a = \frac{7}{4}$.

EXERCISE 4.10. Use the adjoint matrix the find the inverse of the following matrices:

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}.$$

Solution.

$$\begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{pmatrix}^{-1} = \frac{1}{37}\begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & -2 & 5 \end{pmatrix}.$$

EXERCISE 4.11. Use Cramer's rule so solve the following systems:

(1)

$$\begin{aligned} 5x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 4. \end{aligned}$$

(2)

$$\begin{aligned} 2x_1 - 3x_2 &= 2 \\ 4x_1 - 6x_2 + x_3 &= 7 \\ x_1 + 10x_2 &= 1. \end{aligned}$$

Solution.

(1) $x_1 = 1, \; x_2 = -2$.

(2) $x_1 = 1, \; x_2 = 0, \; x_3 = 3$.

CHAPTER 5

Eigenvalues and Eigenvectors

In this chapter we are going to learn a different aspect of matrices which is crucial in the study of linear and nonlinear system of equations. This aspect is the eigenvalues and eigenvectors of square matrices. This chapter contains two parts combined together. The first is dedicated to the basic definition and properties of eigenvalues and eigenvectors, and the second is dedicated to way they are used. Throughout this chapter we will give economically-relevant examples to motivate the mathematical aspects.

5.1. Definition

Let A be an $n \times n$ matrix and let $\mathbf{x} \in \mathbb{R}^n$ be a non-zero vector. We say that $\mathbf{x}$ is an *eigenvector* of A if there exists a value $\lambda \in \mathbb{R}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Similarly, we say that $\lambda \in \mathbb{R}$ is an *eigenvector* of A if there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \lambda\mathbf{x}$.
The equation $A\mathbf{x} = \lambda\mathbf{x}$ could be rewritten as follows:

$$\begin{aligned} A\mathbf{x} &= \lambda I\mathbf{x} \\ A\mathbf{x} - \lambda I\mathbf{x} &= \mathbf{0} \\ (A - \lambda I)\,\mathbf{x} &= \mathbf{0}. \end{aligned}$$

We will usually use the form $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$, for reasons that will later be explained. Let us begin with a basic example.

Example 5.1. Fix $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. We can see that by taking $\lambda = 2$, we get

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

And we can take $\mathbf{x} = (1,0)$ that yields the required result. Thus, $\lambda = 2$ is an eigenvalue with eigenvector $\mathbf{x} = (1,0)$. The same holds for the eigenvalue $\lambda = 3$ with eigenvector $\mathbf{x} = (0,1)$. Note that the eigenvector is determined up to a constant.

Example 5.1 presents a general notion when it comes to the theory of eigenvalues and diagonal matrices. This notion is presented in the following theorem.

Theorem 5.1. *The diagonal entries of a diagonal matrix are its eigenvalues.*

5.2. Properties of eigenvalues and matrices

The first use of eigenvalues is to determine whether a matrix is singular or not.

Theorem 5.2. *A matrix $A \in M_n$ is singular if and only if $\lambda = 0$ is an eigenvalue of A.*

That is, we can use eigenvalues to determine the number of solutions a system of linear equations has. Getting back to the previous representation of the eigenvalues problem, we can see that $(A - \lambda I)\,\mathbf{x} = \mathbf{0}$ is basically a system of linear equations with the matrix $A - \lambda I$. Therefore, we already know that it has a non-trivial solution $\mathbf{x}$ if and only if $\det(A - \lambda I) = 0$. In this case, $A - \lambda I$ is singular and not invertible. The following theorem summarizes these conclusions.

Theorem 5.3. *Let $A \in M_n$ be an $n \times n$ matrix, and let λ be a scalar. The following statements are equivalent:*

(1) $A - \lambda I$ is singular.
(2) $\det(A - \lambda I) = 0$.
(3) $A\mathbf{x} = \lambda\mathbf{x}$ for some non zero vector $\mathbf{x}$.
(4) λ is an eigenvalue of A.

COROLLARY 5.1. *A matrix is invertible if and only if all its eigenvalues are non zero. In addition, if a matrix, with eigenvalues* $\lambda_1, \dots, \lambda_n$, *is invertible, then the eigenvalues of* A^{-1} *are* $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$.

LEMMA 5.1. *Let A be a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$. Then, the eigenvalues of A^k are $\lambda_1^k, \dots, \lambda_n^k$ for any positive integer k.*

These statements show us how closely related are eigenvalues to everything we learned until now. However, the question that still remains unanswered is how to find the eigenvalues of a matrix A? To answer this question, we have the *characteristic polynomial* of A, which is $\det(A - \lambda I) = 0$. This is a polynomial of degree n in λ. Thus, the zeros of this polynomial are the eigenvalues of A. After finding the eigenvalues, we can use the matrix $A - \lambda I$ to compute the eigenvectors.

THEOREM 5.4. *Let $\lambda_1, \dots, \lambda_k$ be k distinct eigenvalues of $A \in M_n$ with corresponding eigenvectors $v_1, \dots, v_k$. Then , $v_1, \dots, v_k$ are linearly independent.*

To conclude, we can summarize previous results and properties in the following manner:

- There exists a non-zero eigenvector $\mathbf{x}$, that is $A\mathbf{x} = \lambda\mathbf{x}$, if and only if $\det(A - \lambda I) = 0$.
- A matrix is invertible if and only if all its eigenvalues are non zero.
- If a matrix with eigenvalues $\lambda_1, \dots, \lambda_n$ is invertible, then the eigenvalues of A^{-1} are $\frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n}$.
- If $\lambda_1, \dots, \lambda_k$ be k distinct eigenvalues with corresponding eigenvectors $v_1, \dots, v_k$, then , $v_1, \dots, v_k$ are linearly independent.

We now turn to compute eigenvalues and eigenvectors using the first equivalence statement.

EXERCISE 5.1. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Solution. We start with the *characteristic polynomial* of A.

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} -1-\lambda & 3 \\ 2 & -\lambda \end{pmatrix} \\ &= \lambda + \lambda^2 - 6 \\ &= (\lambda + 3)(\lambda - 2) \\ \lambda_{1,2} &= 2, -3. \end{aligned}$$

We wish to compute v_1, the eigenvector of $\lambda_1 = 2$.

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We can solve this system as any other system of linear equations and get

$$\begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \xrightarrow{-\frac{1}{3}L_1 \to L_1} \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \xrightarrow{L_2 - 2L_1 \to L_2} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

and we get $x - y = 0$. Thus we can choose $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Similarly, for $v_2$we get

$$\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which means that $x = -\frac{3}{2}y$. Thus, $v_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

EXERCISE 5.2. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

Solution. The eigenvalues are $\lambda_{1,2,3} = 5, 4, 2$. The eigenvectors are

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

EXERCISE 5.3. Find the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}.$$

Solution. The eigenvalues are $\lambda_{1,2,3} = 5, 4, -1$. The eigenvectors are

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

EXERCISE 5.4. Find the eigenvalues and eigenvectors of the following matrices

$$\begin{aligned} A &= \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}, \\ B &= \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}, \\ D &= \begin{pmatrix} 0 & 0 & -2 \\ 0 & 7 & 0 \\ 1 & 0 & -3 \end{pmatrix}. \end{aligned}$$

Solution. The eigenvalues and eigenvectors of the matrices are:

- A: The eigenvalues are $\lambda_{1,2} = 3, 5$. The eigenvectors are
$$v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
- B: The eigenvalues are $\lambda_{1,2} = 2, 1$. The eigenvectors are
$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$
- C: The eigenvalues are $\lambda_{1,2} = -1, -2$. The eigenvectors are
$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- A: The eigenvalues are $\lambda_{1,2,3} = 7, -1, -2$. The eigenvectors are

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \; v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

EXERCISE 5.5. Prove that the eigenvalues of a triangular matrix are its diagonal entries.

Solution. Writing down the *characteristic polynomial* of a triangular matrix D gives

$$\det(D - \lambda I) = \prod_{i=1}^{n} (d_{ii} - r) = 0.$$

The only solution in this case is $\lambda_i = d_{ii}$ for every i, as required.

5.2.1. Trace of a matrix.

The *trace* of a matrix A is the sum of its diagonal entries. That is,

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}.$$

The following theorem shows that the trace and the determinant are directly related to the sum of the eigenvalues of its matrix.

THEOREM 5.5. *Let $A \in M_n$ be a matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Then,*

$$\text{trace}(A) = \sum_{i=1}^{n} \lambda_i, \;\; \det(A) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

This theorem is very useful when trying to find the determinant of a matrix, and even when trying to verify the computation of the eigenvalues or determinant.

5.3. Applications

5.3.1. Difference equations.

Eigenvalues and eigenvectors are very useful when it comes to solving dynamical problems modeled through linear difference equation. We begin with a very simple example that illustrates what a linear difference equations are. Assume you have x_0 dollars deposit that gain an interest rate of r each year. How much money will you have after n years? Denote this amount by x_n. The amount increases by a factor of $1 + r$ on an annual basis. The difference equation in this case is

$$x_{k+1} = (1 + r) x_k.$$

Thus, as you probably have guessed, the solution is $x_n = (1 + r)^n x_0$.
This is basically a very simple example. Now what happens if we have two variable x_k, y_k such that their dynamics are connected. That is,

$$\begin{aligned} x_{k+1} &= a x_k + b y_k \\ y_{k+1} &= c x_k + d y_k. \end{aligned}$$

How do we solve this problem? Since they both depend on each other, we need to solve them simultaneously, which could be quite difficult. Remember, our goal is to find a formula

$$\begin{aligned} x_n &= f(x_0, y_0, n, a, b, c, d) \\ y_n &= g(x_0, y_0, n, a, b, c, d). \end{aligned}$$

In the case that $b = c = 0$, then the equations are uncoupled and we can solve them separately. For that purpose, we use eigenvalues and eigenvectors. Let us present the coupled equations in the following form:

$$\begin{aligned} \mathbf{z}_{k+1} &= \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} = A\mathbf{z}_k \\ \mathbf{z}_{k+1} &= A\mathbf{z}_k. \end{aligned}$$

We are going to survey to ways to deal with this problem. Both are quite similar and are based on eigenvalues and eigenvectors.

5.3.1.1. *The powers of a matrix.*

We wish to solve the difference equations $\mathbf{z}_{k+1} = A\mathbf{z}_k$ given the initial conditions $\mathbf{z}_0$. Therefore, we know that

$$\begin{aligned} \mathbf{z}_1 &= A\mathbf{z}_0 \\ \mathbf{z}_2 &= A\mathbf{z}_1 = AA\mathbf{z}_0 = A^2 z_0 \\ \mathbf{z}_3 &= A^3\mathbf{z}_0 \\ &\vdots \\ \mathbf{z}_k &= A^k\mathbf{z}_0. \end{aligned}$$

In general, it is difficult to calculate A^k unless it is diagonal. If it is diagonal, then A^k is similar to A when all the diagonal entries are taken with the power k. That is, if

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix},$$

then

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n^k \end{pmatrix}.$$

clearly, A is not a diagonal matrix (if it was, we would have uncoupled difference equations). However, if we could find a non-singular matrix P such that $P^{-1}AP = D$ when D is the matrix described above, then

$$\begin{aligned} A^k &= \left(PDP^{-1}\right)\left(PDP^{-1}\right)\cdots\left(PDP^{-1}\right) \\ &= PD\left(P^{-1}P\right)D\left(P^{-1}P\right)D\left(P^{-1}P\right)D\cdots\left(P^{-1}P\right)DP^{-1} \\ &= PDIDIDID\cdots IDP^{-1} \\ &= PD^kP^{-1}. \end{aligned}$$

Hence, if D is a diagonal matrix and P is invertible, then A^k is easily computable. The following theorem explains how P and D are found.

THEOREM 5.6. *Let $A \in M_n$ be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors $v_1, \ldots, v_n$. Define*

$$P = [v_1 \; v_2 \; \cdots \; v_n].$$

If P^{-1} exists, then

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix} P^{-1},$$

and

$$A^k = P \begin{pmatrix} \lambda_1^k & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n^k \end{pmatrix} P^{-1}.$$

Thus, the solution of the corresponding system of difference equations $\mathbf{z}_{k+1} = A\mathbf{z}_k$ with initial vector $\mathbf{z}_0$ is

$$\mathbf{z}_k = P \begin{pmatrix} \lambda_1^k & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n^k \end{pmatrix} P^{-1}\mathbf{z}_0.$$

DEFINITION 5.1. Let $A \in M_n$ be a square matrix.

- A is *diagonalizable* if there exists a non-singular matrix P such that $P^{-1}AP = D$ when D is a diagonal matrix.
- A is orthogonal if $A^{-1} = A^T$.

PROPOSITION 5.1. *If $A \in M_n$ is a symmetric matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding normalized*[1] *eigenvectors $v_1, \ldots, v_n$, there exists an orthogonal matrix P such that*

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix},$$

when $P = [v_1 \ v_2 \ \cdots \ v_n]$

LEMMA 5.2. *If a matrix $A \in M_n$ has n distinct eigenvalues, then it is diagonalizable. Moreover, A is diagonalizable if and only if it has n independent eigenvectors.*

EXERCISE 5.6. For each matrix, find an orthogonal matrix that diagonalizes it.

(1)
$$\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix},$$

(2)
$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix},$$

(3)
$$\begin{pmatrix} 0.6 & 0.4 \\ 0.4 & 0.6 \end{pmatrix},$$

[1] We say that a vector v is normalized if $v^T v = 1$. That is, the sum of its squared coordinates equals 1.

(4)

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

(5)

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

(6)

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

Solution.

(1) Eigenvalues are $\lambda = -2, 6$. Normalized eigenvectors are

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \ v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The orthogonal matrix is

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(2) Eigenvalues are $\lambda = 0, 5$. Normalized eigenvectors are

$$v_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \ v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The orthogonal matrix is

$$P = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}.$$

(3) The orthogonal matrix is

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(4) The orthogonal matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

(5) The orthogonal matrix is

$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

(6) The orthogonal matrix is

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

Exercise 5.7. For each matrix, find an orthogonal matrix that diagonalizes it.

(1)

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

(2)
$$\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},$$

(3)
$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Solution.

(1) The orthogonal matrix is
$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(2) The orthogonal matrix is
$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

(3) The orthogonal matrix is
$$P = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}.$$

Let us discuss a concrete example, and then go over to the next method.

EXAMPLE 5.2. **The Leslie Population model.**
The mathematical demographer P.H. Leslie, introduced the model named after him in 1945, to describe how the population evolves (either grows or shrinks).
Consider an organism that lives for two years. The organism can reproduce in the first year and the second year as well. In addition, the organism can die in the first year. Define the following parameters:

- b_i where $i = 1, 2$, is the birth rate of individuals in their i^{th} year.
- d_1 is the death rate of first year individuals.
- x_k, y_k are the number of first-year individuals and second-year individuals on year k, respectively.

The dynamics of this population are described in the following difference equations:
$$\begin{aligned} x_{k+1} &= b_1 x_k + b_2 y_k \\ y_{k+1} &= (1 - d_1)\, x_k. \end{aligned}$$

Our main goal is to solve this problem. For simplicity, we will use specific numbers.
$$\begin{aligned} x_{k+1} &= x_k + 4y_k \\ y_{nk+1} &= 0.5 x_k, \end{aligned}$$

or, in its matrix form,
$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0.5 & 0 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}.$$

Fix $A=\begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}$ and let us find the eigenvalues and eigenvectors of A.

$$\begin{aligned}\det\left(A-I\lambda\right) &= \det\begin{pmatrix}1-\lambda & 4\\ 0.5 & -\lambda\end{pmatrix}\\ &= -\lambda\left(1-\lambda\right)-2=0\\ \lambda^2-\lambda-2 &= 0\\ \left(\lambda-2\right)\left(\lambda+1\right) &= 0\\ \lambda_{1,2} &= 2,-1.\end{aligned}$$

The eigenvector $v_1$is

$$\begin{aligned}\begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} &= 2\begin{pmatrix}x\\ y\end{pmatrix}\\ v_1 &= \begin{pmatrix}4\\ 1\end{pmatrix},\end{aligned}$$

and $v_2=\begin{pmatrix}-2\\ 1\end{pmatrix}$. Define $P=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}$. Its inverse matrix is $P^{-1}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}$ (verify this!). Note that

$$\begin{aligned}PDP^{-1} &= \begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\\ &= \begin{pmatrix}8 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\\ &= \begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}=A,\end{aligned}$$

as required. Thus,

$$\begin{aligned}\mathbf{z}_k &= \begin{pmatrix}x_k\\ y_k\end{pmatrix}\\ &= P\begin{pmatrix}2^k & 0\\ 0 & (-1)^k\end{pmatrix}P^{-1}\mathbf{z}_0\\ &= \begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}2^k & 0\\ 0 & (-1)^k\end{pmatrix}\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}\\ &= \begin{pmatrix}\frac{2^{k+1}+(-1)^k}{3} & \frac{2^{k+2}+4(-1)^{k+1}}{3}\\ \frac{2^k+(-1)^{k+1}}{6} & \frac{2^k+2(-1)^k}{3}\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix},\end{aligned}$$

and the problem is solved.

REMARK 5.1. One can notice an interesting situation when $|\lambda_i|<1$ for every $i=1,\ldots,n$. If that is indeed the case, then $\lambda_i^n\to 0$ as $n\to\infty$. This implies, that whenever the absolute value of all the eigenvalues is less than 1, the general solution tends to $\mathbf{0}$. This situation is called as *asymptotically stable*, since the system will remain in $\mathbf{0}$ once it is reached.

5.3.1.2. *Coordinates transformations.*

The second method aims at transforming the coordinates of the problem. Assume that we can transform the coordinates of $\mathbf{z}_k$ to different coordinates, denoted $\mathbf{Z}_k$, by the matrices P and P^{-1} such that

$$\mathbf{z}_k=P\mathbf{Z}_k,\ \mathbf{Z}_k=P^{-1}\mathbf{z}_k.$$

In that case,

$$\begin{aligned}\mathbf{Z}_{k+1} &= P^{-1}\mathbf{z}_{k+1}\\ &= P^{-1}\left(A\mathbf{z}_k\right)\\ &= \left(P^{-1}A\right)\mathbf{z}_k\\ &= \left(P^{-1}A\right)P\mathbf{Z}_k\\ &= \left(P^{-1}AP\right)\mathbf{Z}_k,\end{aligned}$$

and if $P^{-1}AP$ is a diagonal matrix, then we get a system of uncoupled equations, where we can solve each separately. However, how could we find such matrices? The answer is as before, use eigenvectors! Note that we have the following equivalent problem

$$P^{-1}AP = D,$$

or equivalently,

$$AP = PD,$$

when we require D to be a diagonal matrix. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the $n \times n$ matrix A, whose eigenvectors are $v_1, \ldots, v_n$. Define the matrix P to be a matrix whose columns are the eigenvectors $v_1, \ldots, v_n$ such that

$$P = \left[v_1\ v_2\ \cdots\ v_n\right],$$

and define D to be a diagonal matrix whose diagonal entries are the respected eigenvalues $\lambda_1, \ldots, \lambda_n$ such that

$$D = \begin{pmatrix}\lambda_1 & 0 & \cdots & \cdots & 0\\ 0 & \lambda_2 & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \lambda_n\end{pmatrix}.$$

Now, we can see that

$$\begin{aligned}AP &= A\left[v_1\ v_2\ \cdots\ v_n\right]\\ &= \left[Av_1\ Av_2\ \cdots\ Av_n\right]\\ &= \left[\lambda_1 v_1\ \lambda_2 v_2\ \cdots\ \lambda_n v_n\right]\\ &= \left[\lambda_1 v_1\ \lambda_2 v_2\ \cdots\ \lambda_n v_n\right],\end{aligned}$$

and

$$\begin{aligned}PD &= \left[v_1\ v_2\ \cdots\ v_n\right]\begin{pmatrix}\lambda_1 & 0 & \cdots & \cdots & 0\\ 0 & \lambda_2 & 0 & \cdots & 0\\ \vdots & 0 & \ddots & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & 0\\ 0 & 0 & 0 & 0 & \lambda_n\end{pmatrix}\\ &= \left[\lambda_1 v_1\ \lambda_2 v_2\ \cdots\ \lambda_n v_n\right] = AP,\end{aligned}$$

and we get the required result. Therefore, we can use the eigenvectors and eigenvalues to create P and D such that the problem becomes uncoupled.

EXAMPLE 5.3. **The Leslie Population model (revisited).** We use the second method to solve again the same problem. Remember that the dynamics of the problem are given by

$$\begin{pmatrix}x_{n+1}\\ y_{n+1}\end{pmatrix} = \begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}\begin{pmatrix}x_k\\ y_k\end{pmatrix}.$$

Using the previously defined matrices: $P=\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}$ and its inverse $P^{-1}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}$ and the new coordinates are

$$\begin{pmatrix}X_{n+1}\\ Y_{n+1}\end{pmatrix}=\begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}x_{n+1}\\ x_{n+1}\end{pmatrix}.$$

Thus, we get

$$\begin{aligned}
\begin{pmatrix}X_{n+1}\\ Y_{n+1}\end{pmatrix} &= \begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}x_{n+1}\\ x_{n+1}\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}\begin{pmatrix}x_n\\ y_n\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}1 & 4\\ 0.5 & 0\end{pmatrix}\begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}X_n\\ Y_n\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}8 & 2\\ 2 & -1\end{pmatrix}\begin{pmatrix}X_n\\ Y_n\end{pmatrix}\\
&= \begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}X_n\\ Y_n\end{pmatrix},
\end{aligned}$$

and we got the equations uncoupled.

We can solve each and get

$$\begin{aligned}
X_{n+1} &= 2X_n\\
Y_{n+1} &= -Y_n\\
&\Downarrow\\
X_n &= 2^n X_0\\
Y_n &= (-1)^n XY_0
\end{aligned}$$

when X_0, Y_0 are values that we can get from the initial conditions. Hence,

$$\begin{aligned}
\begin{pmatrix}x_n\\ y_n\end{pmatrix} &= \begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}X_n\\ Y_n\end{pmatrix}\\
&= \begin{pmatrix}4 & -2\\ 1 & 1\end{pmatrix}\begin{pmatrix}2^n X_0\\ (-1)^n Y_0\end{pmatrix},
\end{aligned}$$

when

$$\begin{aligned}
\begin{pmatrix}X_0\\ Y_0\end{pmatrix} &= P^{-1}\begin{pmatrix}x_0\\ y_0\end{pmatrix}\\
&= \begin{pmatrix}\frac{1}{6} & \frac{1}{3}\\ -\frac{1}{6} & \frac{2}{3}\end{pmatrix}\begin{pmatrix}x_0\\ y_0\end{pmatrix}.
\end{aligned}$$

The following theorems summarize these conclusions and observations.

THEOREM 5.7. *Let $A \in M_n$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors $v_1, \ldots, v_n$. Define*

$$P = [v_1\ v_2\ \cdots\ v_n].$$

If P^{-1}exists, then

$$P^{-1}AP = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & \lambda_n \end{pmatrix}.$$

Conversely, if $P^{-1}AP == D$ when D is a diagonal matrix, then the columns of P are the eigenvectors of A and the diagonal entries of D are the eigenvalues of A.

The following theorem gives the general solution for general difference equations.

THEOREM 5.8. *Let $A \in M_n$ be a matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and the corresponding eigenvectors $v_1, \ldots, v_n$. The general solution for the system of difference equations $\mathbf{z}_{n+1} = A\mathbf{z}_n$ is*

$$\mathbf{z}_n = \sum_{i=1}^{n} c_i r_i^n v_i,$$

where c_i are given by the initial conditions.

EXERCISE 5.8. For each of the following matrices, find a non-singular matrix P and a diagonal matrix D so that $D = P^{-1}AP$.

(1)
$$\begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix},$$

(2)
$$\begin{pmatrix} 1 & -1 \\ 3 & 4 \end{pmatrix},$$

(3)
$$\begin{pmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix},$$

(4)
$$\begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Solution.

(1)
$$P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

(2)
$$P = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}.$$

(3)
$$P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(4)
$$P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

EXERCISE 5.9. Find the general solution of the following systems of difference equations:

(1)

$$\begin{aligned} x_{n+1} &= 3x_n \\ y_{n+1} &= x_n + 2y_n; \end{aligned}$$

(2)

$$\begin{aligned} x_{n+1} &= y_n \\ y_{n+1} &= -x_n + 5y_n; \end{aligned}$$

(3)

$$\begin{aligned} x_{n+1} &= x_n - y_n \\ y_{n+1} &= 2x_n + 4y_n. \end{aligned}$$

Solution.

(1)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = x_0 \cdot 3^n \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + y_0 \cdot 2^n \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(2)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = x_0 \cdot \left(\frac{5+\sqrt{21}}{2}\right)^n \cdot \begin{pmatrix} 1 \\ \frac{5+\sqrt{21}}{2} \end{pmatrix} + y_0 \cdot \left(\frac{5-\sqrt{21}}{2}\right)^n \cdot \begin{pmatrix} 1 \\ \frac{5-\sqrt{21}}{2} \end{pmatrix}.$$

(3)

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = x_0 \cdot 2^n \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + y_0 \cdot 3^n \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

CHAPTER 6

Quadratic forms

6.1. Introduction to quadratic forms

The natural point to start discussing optimization problems is linear problems. Since we know how to solve linear problems relatively easily, we can now move forward with the next step, which is quadratic forms. The simplest case is taught in high school through the quadratic formula. But what happens when we have more than one variable? In that case, we need to generalize our technique, and matrices we be much handy for that.

First, let us present the problem. A quadratic form $Q(x_1, \ldots, x_n)$ in n variables $x_1, x_2, \ldots, x_n$ is a polynomial expression in which each component term has a degree two (i.e. each term is a product of x_i and x_j , where $i, j = 1, 2, \cdots, n$). That is, $Q(x_1, \ldots, x_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j$ where $a_{ij} \in \mathbb{R}$.

Claim 6.1. Each quadratic form $Q(x_1, \ldots, x_n)$ in n variables $x_1, x_2, \ldots, x_n$ can be represented by a symmetric matrix A so that

$$Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x},$$

where $\mathbf{x}^T = (x_1, \ldots, x_n)$.

For example, the quadratic form $Q(x_1, x_2) = a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$ can be written as

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} \\ \frac{1}{2}a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

For the case of $n = 3$: $Q(x_1, x_2, x_3) = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{23}x_2x_3$ can be written as

$$\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} a_{11} & \frac{1}{2}a_{12} & \frac{1}{2}a_{13} \\ \frac{1}{2}a_{12} & a_{22} & \frac{1}{2}a_{23} \\ \frac{1}{2}a_{13} & \frac{1}{2}a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

To understand the importance of these forms, let us consider a few concrete examples.

Example 6.1. A manager faces an optimization problem. The revenue from producing x_1 units of product A and x_2 units of product B is $R(x_1, x_2) = 8x_1x_2$. However, the cost is given by $C(x_1, x_2) = 3x_1^2 + 6x_2^2$. How much should the manager produce? Use quadratic forms to present the problem.

Solution. One can formulate the problem as $f(x_1, x_2) = 8x_1x_2 - 3x_1^2 - 6x_2^2$, or equivalently

$$f(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

In the following section, we will see that in such a case, the optimal production is $(0, 0)$.

6.2. Definiteness of quadratic forms

There is an easy way to distinguish between different types of quadratic forms, using its sign.

Definition 6.1. **(Definite quadratic forms)** Let $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ be a quadratic form.

- If $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, then A is called *positive definite*.
- If $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then A is called *positive semi-definite*.
- If $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, then A is called *negative definite*.
- If $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$, then A is called *negative semi-definite*.

- Otherwise A is called *indefinite*.

Why is this distinction so important? If a quadratic form is either positive or negative, we know that it could be minimized or maximized in $\mathbf{x} = \mathbf{0}$. For this reason, optimizing such a form is relatively easy. However, determining whether a quadratic form is positive or negative is not easy, as it requires to know the sign of the form for every vector $\mathbf{x}$.
To simplify the classification, we require an additional definition.

DEFINITION 6.2. **(Principal minors)** Let $A \in M_n$ be an $n \times n$ matrix. The $k \times k$ matrix formed from A by deleting the $n-k$ last rows and the last $n-k$ last columns is called a k^{th} order *leading principal sub-matrix* of A. That is, if

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

then the k^{th} order leading principal sub-matrix of A is

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix}.$$

The determinant of the k^{th} order leading principal sub-matrix of A is called the k^{th} order *leading principal minor* (LPM) of A, and it is denoted by A_k. We use the leading principal minors in the following characterization of definite quadratic forms.

THEOREM 6.1. *Let A be an $n \times n$ matrix.*

- A is positive definite if and only if all its n leading principal minors are positive.
- A is positive semi-definite if and only if all its n leading principal minors are non-negative.
- A is negative definite if and only if its n leading principal minors follow the rule $(-1)^k \cdot A_k > 0$ for every $k = 1, \ldots, n$.
- A is negative semi-definite if and only if its n leading principal minors follow the rule $(-1)^k \cdot A_k \geq 0$ for every $k = 1, \ldots, n$.

EXERCISE 6.1. Find the definiteness of the following matrices:

$$\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Solution.

(1) Considering $\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}$. Since $|A_1| = 2$ and $|A_2| = 14 - 9 = 5$, the matrix is positive definite.
(2) Considering $\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$. Since $|A_1| = 2$ and $|A_2| = 14 - 16 = -2$, the matrix is indefinite.
(3) Considering $\begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$. Since $|A_1| = 0$ and $|A_2| = 0$, the matrix could be either positive semi-definite or negative semi-definite. When writing down the form explicitly, we get $Q(x_1 x_2) = cx_2^2$, and the sign of c clearly determines the definiteness of the matrix.

EXERCISE 6.2. Determine the definiteness of the following symmetric matrices:

(1)
$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

(2)
$$\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}.$$
(3)
$$\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}.$$
(4)
$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}.$$
(5)
$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 5 \\ 0 & 5 & 6 \end{pmatrix}.$$
(6)
$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$
(7)
$$\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 2 & 0 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 5 & 0 & 6 \end{pmatrix}.$$

Solution.

(1) The LPMs are 2 and 1, thus the matrix is positive definite.
(2) The LPMs are -3 and -1, thus the matrix is indefinite.
(3) The LPMs are -3 and 2, thus the matrix is negative definite.
(4) The LPMs are 2 and 0, thus the matrix is positive semi-definite.
(5) The LPMs are $1, 0$, and -25, thus the matrix is indefinite.
(6) The LPMs are $-1, 0$, and 0, thus the matrix is negative semi-definite.
(7) The first three LPMs are $1, 2$, and -10, thus the matrix is indefinite.

EXERCISE 6.3. Assume that A, B are positive definite matrices. Prove that $A + B$ is also a positive definite matrix.

Solution. If $\mathbf{x}^T A\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$, and $\mathbf{x}^T B\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$, then
$$\mathbf{x}^T (A + B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0,$$
as well.

EXERCISE 6.4. Let $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form where A is symmetric. Prove that a necessary condition for A to be positive definite is that all its diagonal entries are positive. Give an example to show that this necessary condition is not a sufficient condition.

Solution. Suppose $Q(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$. For $x = e_i = (0, \dots, 0, 1, 0, \dots) = a_{ii} > 0$. If A is positive semi-definite, we must have $a_{ii} = \mathbf{e_i}^T A\mathbf{e_i} \geq 0$. Similarly, if A was negative definite. To show that these conditions are not sufficient consider the indefinite matrix $\begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$.

6.2.1. Definiteness of quadratic forms.

The eigenvalues have a strong connection to the definiteness of quadratic forms.

THEOREM 6.2. *Let A be a symmetric matrix. Then,*

- A is positive definite (respectively, semi-definite) if and only if all its eigenvalues are positive (respectively, non negative).
- A is negative definite (respectively, semi-definite) if and only if all its eigenvalues are negative (respectively, non positive).
- A is indefinite if and only if its has at least one positive eigenvalue and at least one negative eigenvalue.

The following theorem regarding positive definite matrices is important in statistics and econometrics.

THEOREM 6.3. *Let A be a symmetric matrix. Then, the following statements are equivalent:*

- A is positive definite.
- There exists a non-singular matrix B such that $A = B^T B$.
- There exists a non-singular matrix Q such that $Q^T A Q = I$.

EXERCISE 6.5. Find the definiteness of the following matrices:

$$\begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}.$$

Solution.

(1) The eigenvalues are both positive, thus the matrix is positive definite.
(2) The eigenvalues are $\lambda_1 < 0 < \lambda_2$, thus the matrix is indefinite.
(3) The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = c$, and the sign of c clearly determines the definiteness of the matrix.

EXERCISE 6.6. Determine the definiteness of the following symmetric matrices:

(1)
$$\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$
(2)
$$\begin{pmatrix} -3 & 4 \\ 4 & -5 \end{pmatrix}.$$
(3)
$$\begin{pmatrix} -3 & 4 \\ 4 & -6 \end{pmatrix}.$$
(4)
$$\begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix}.$$

Solution.

(1) The eigenvalues are 3 and 1, thus the matrix is positive definite.
(2) The eigenvalues are $\lambda_1 < 0 < \lambda_2$, thus the matrix is indefinite.
(3) The eigenvalues are both negative, thus the matrix is negative definite.
(4) The eigenvalues are 10 and 0, thus the matrix is positive semi-definite.

6.3. Linear constraints and bordered matrices

Clearly, one can use the classification of quadratic forms in order to determine a global maxima or minima. However, usually in economics, we are subjected to constraints. Meaning that the optimization is not taken with respect to $\mathbb{R}^n$, but only s part of it. What happens than? we start with a simple example.

EXAMPLE 6.2. Let $Q(x_1, x_2) = x_1^2 - x_2^2$ be a quadratic form. $Q(x_1, x_2)$ is indefinite as we can see that the origin is not a maxima or minima. However, what happens if we impose the constraint that $x_2 = 0$?

In this case, $Q(x_1, 0) = x_1^2$ which has a strict global minimum in $x_1 = 0$. Therefore, when we restrict out attention to specific set, the quadratic form can transfer from definite to indefinite and vice versa.

THEOREM 6.4. *Fix* $Q(x_1, x_2) = \mathbf{x}^T A \mathbf{x}$ *where* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *and assume that the constraint set is* $B_1 x_1 + B_2 x_2 = 0$. *The quadratic form given the constraint is positive (respectively, negative) if and only if*

$$\det \begin{pmatrix} 0 & B_1 & B_2 \\ B_1 & a & b \\ B_2 & b & c \end{pmatrix}$$

is negative (respectively, positive). The new matrix is called the bordered matrix.

In other words, we take the linear constraint and the original matrix and generate a new matrix which determines the definiteness of the restricted quadratic form.
For the general case, we have the following theorem.

THEOREM 6.5. *Let* $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ *be a quadratic form with* n *variables and let* $B\mathbf{x} = \mathbf{0}$ *be a set of linear constraints where*

$$B = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{pmatrix}.$$

Define the bordered matrix

$$H = \begin{pmatrix} \mathbf{0} & B \\ B^T & A \end{pmatrix}.$$

- If the sign of $\det(H)$ equals $(-1)^n$ and if the last $n-m$ leading principal minors of H alternate in sign, then Q is negative definite.
- If the sign of $\det(H)$ and the signs of the last $n-m$ leading principal minors of H equal $(-1)^m$, then Q is positive definite.

EXERCISE 6.7. Check the definiteness of $Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$ on the constraint set

$$x_2 + x_3 + x_4 = 0, \; x_1 - 9x_2 + x_4 = 0.$$

Solution. First we form the bordered matrix

$$H_6 = \begin{pmatrix} 0 & 0 & | & 0 & 1 & 1 & 1 \\ 0 & 0 & | & 1 & -9 & 0 & 1 \\ - & - & - & - & - & - & - \\ 0 & 1 & | & 1 & 0 & 0 & -1 \\ 1 & -9 & | & 0 & -1 & 2 & 0 \\ 1 & 0 & | & 0 & 2 & 1 & 0 \\ 1 & 1 & | & -1 & 0 & 0 & 1 \end{pmatrix}.$$

Since the problem has $n = 4$ variables and $m = 2$ constraints, we need to check the largest $n - m = 2$ LPMs which are $\det(H_6)$ and

$$\det(H_5) = \det \begin{pmatrix} 0 & 0 & | & 0 & 1 & 1 \\ 0 & 0 & | & 1 & -9 & 0 \\ - & - & - & - & - & - \\ 0 & 1 & | & 1 & 0 & 0 \\ 1 & -9 & | & 0 & -1 & 2 \\ 1 & 0 & | & 0 & 2 & 1 \end{pmatrix}.$$

Note that $(-1)^m = 1$. In fact, $\det(H_6) = 24$ and $\det(H_5) = 77$, so Q is positive definite on the given constraint set.

EXERCISE 6.8. Determine the definiteness of the following constrained quadratics:

(1) $Q(x_1, x_2) = x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.
(2) $Q(x_1, x_2) = 4x_1^2 + 2x_1x_2 - x_2^2$, subject to $x_1 + x_2 = 0$.
(3) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.
(4) $Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 4x_1x_3 - 2x_1x_2$, subject to $x_1 + x_2 + x_3 = 0$ and $x_1 + x_2 - x_3 = 0$.
(5) $Q(x_1, x_2, x_3) = x_1^2 - x_3^2 + 4x_1x_2 - 6x_2x_3$, subject to $x_1 + x_2 - x_3 = 0$.

Solution.

(1) $n = 2, m = 1$. The bordered matrix is

$$H = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 1 & 1 \\ 1 & 1 & -1 \end{array}\right).$$

$\det(H) = 2 > 0$, negative definite.

(2) $n = 2, m = 1$. The bordered matrix is

$$H = \left(\begin{array}{c|cc} 0 & 1 & 1 \\ \hline 1 & 4 & 1 \\ 1 & 1 & -1 \end{array}\right).$$

$\det(H) = -1 < 0$, positive definite.

(3) $n = 3, m = 2$. The bordered matrix is

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & -1 \end{array}\right).$$

$\det(H) = 16 > 0$, positive definite.

(4) $n = 3, m = 2$. The bordered matrix is

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & -1 & 2 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 2 & 0 & 1 \end{array}\right).$$

$\det(H) = 16 > 0$, positive definite.

(5) $n = 3, m = 1$. The bordered matrix is

$$H = \left(\begin{array}{cc|ccc} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 1 & 1 & 2 & -3 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & -1 & -3 & 0 & -1 \end{array}\right).$$

$\det(H) = 4 > 0$ and the determinant of the next LPM is 3, so the matrix is indefinite.

Part 3

Advanced calculus

CHAPTER 7

Calculus of several variables

7.1. Vectors and norms in $\mathbb{R}^n$.

7.1.1. Inner products and norms.

After reviewing one-variable calculus, we now turn to calculus in several variables. Before we discuss functions, limits and derivatives in several variables, we survey a few basic properties in the Euclidean space $\mathbb{R}^n$ when $n \in \mathbb{N}$.

The basic elements of $\mathbb{R}^n$ are vectors with n coordinates. Between these objects we can do almost every arithmetic we did in $\mathbb{R}$. For example, let $v = (v_1, \ldots, v_n)$ and $u = (u_1, \ldots, u_n)$ be two vectors in $\mathbb{R}^n$, then

(1) Addition: $u + v = (u_1 + v_1, \ldots, u_n + v_n)$.
(2) Subtraction: $u - v = (u_1 - v_1, \ldots, u_n - v_n)$, which is identical to addition of u and $-v$.
(3) Multiplication by a scalar: $cv = (cv_1, \ldots, cv_n)$, when $c \in \mathbb{R}$.

Other actions are more problematic, such as deviating, and are usually not well-defined. The first new action we discuss is taking the inner product of two vectors. The inner product $u \cdot v$ of u and v is (sometimes denoted $\langle u, v\rangle$)

$$u \cdot v = \sum_{i=1}^{n} u_i v_i.$$

The inner product could be quite useful in economic problems. E.g., assume that a firm uses n inputs with values $x_1, \ldots, x_n$. The price per unit for x_i is p_i. Thus, the total cost of production is $p_1x_1 + \cdots + p_nx_n$, which could be written as $\mathbf{p} \cdot \mathbf{x}$, when $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{x} = (x_1, \ldots, x_n)$.

The next element we need to focus on is the concept of length and distance in $\mathbb{R}^n$. The concept of length in these spaces is described by *a norm.* As there a many types of norms, we usually deal with the *Euclidean norm*, also know as the L^2 *norm* or the L^2 *distance*, and it is defined as follows:

$$||u|| = \sqrt{\sum_{i=1}^{n} u_i^2}.$$

Clearly, $||u|| = \sqrt{u \cdot u}$. Using this notion, the distance between two vectors u and v is given by

$$||u - v|| = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}.$$

The Euclidean norm, as well as other norms, have the following properties:

- Non negative: $||u|| \geq 0$ for every vector u, and $||u|| = 0$ if and only if $u = (0, \ldots, 0)$.
- For every $c \in \mathbb{R}$ and every vector v, it holds that $||cu|| = |c| \cdot ||u||$.
- The triangle inequality. For every two vectors u, v, it holds that $||u + v|| \leq ||u|| + ||v||$.

The inner product could also be presented through the angle between the two vectors. Assume that θ is the angle between u and v, then $u \cdot v = \|u\| \|v\| \cos(\theta)$.

Two vector u, v are *orthogonal*, or perpendicular, if the inner product (sometimes referred to as *the scalar product*) is zero. That is, if $u \cdot v = 0$. In terms of the angle between the vectors, it follows that $\theta = \pi/2$, hence perpendicular. Moreover, if, in addition, $||u|| = ||v|| = 1$, then the vectors are *orthonormal.* Note that every vector $u \neq 0$, can be normalized by taking $\bar{u} = u/||u||$.

EXERCISE 7.1. Let u, v, w be three vectors in $\mathbb{R}^n$ and let c be a real number. Prove the following statements:

(1) $u \cdot (v + w) = u \cdot v + u \cdot w$.
(2) $u \cdot (rv) = (ru) \cdot v$.
(3) $u \cdot u \geq 0$.

Solution. All the proofs are based on algebraic manipulations.

(1) We are able to split the relevant sums because they are finite.

$$\begin{aligned} u \cdot (v+w) &= u \cdot (v_1 + w_1, \ldots, v_n + w_n) \\ &= \sum_{i=1}^{n} u_i (v_i + w_i) \\ &= \sum_{i=1}^{n} (u_i v_i + u_i w_i) \\ &= \sum_{i=1}^{n} u_i v_i + \sum_{i=1}^{n} u_i w_i \\ &= u \cdot v + u \cdot w. \end{aligned}$$

(2) We use the fact that r is a real number, not a vector, and get

$$\begin{aligned} u \cdot (rv) &= u \cdot (rv_1, \ldots, rv_n) \\ &= \sum_{i=1}^{n} u_i (rv_i) \\ &= \sum_{i=1}^{n} (ru_i) v_i \\ &= (ru_1, \ldots, ru_n) \cdot v \\ &= (ru) \cdot v. \end{aligned}$$

(3) Note that for every $u_i \in \mathbb{R}$, it holds that $u_i^2 \geq 0$. Thus,

$$u \cdot u = \sum_{i=1}^{n} u_i^2 \geq 0,$$

as the sum of non-negative numbers.

EXERCISE 7.2. For any two vector $u, v \in \mathbb{R}^n$, prove that $||u|| - ||v||| \leq ||u - v||$.

PROOF. The proof is based on the triangle inequality of the Euclidean norm, which states that for every two vectors x, y, it holds that $||x + y|| \leq ||x|| + ||y||$. Fix $x = u - v$ and $y = v$ to get

$$\begin{aligned} ||x+y|| &\leq ||x|| + ||y|| \\ &\Downarrow \\ ||u - v + v|| &\leq ||u - v|| + ||v|| \\ &\Downarrow \\ ||u|| - ||v|| &\leq ||u - v||. \end{aligned}$$

Now fix $y = v - u$ and $x = u$ to get

$$\begin{aligned} ||x+y|| &\leq ||x|| + ||y|| \\ &\Downarrow \\ ||u+v-u|| &\leq ||u|| + ||v-u|| \\ &\Downarrow \\ ||v|| - ||u|| &\leq ||u-v||. \end{aligned}$$

Therefore,

$$-||u-v|| \leq ||u|| - ||v|| \leq ||u-v||,$$

as needed. □

EXERCISE 7.3. For each of the following vectors, find a normalized vector that point in the same direction.

(1) $(3,4)$.
(2) $(6,0)$.
(3) $(1,1,1)$.
(4) $(-1,2,-3)$.

Solution. To normalize a vector $v \neq 0$, all we need to do is to multiply it by a factor $1/||v||$.

(1) $(3,4)\,\frac{1}{||(3,4)||} = (3,4)\,\frac{1}{\sqrt{9+15}} = \left(\frac{3}{5}, \frac{4}{5}\right)$.
(2) $(6,0)\,\frac{1}{||(6,0)||} = (6,0)\,\frac{1}{\sqrt{36+0}} = (1,0)$.
(3) $(1,1,1)\,\frac{1}{||(1,1,1)||} = (1,1,1)\,\frac{1}{\sqrt{1+1+1}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.
(4) $(-1,2,-3)\,\frac{1}{||(-1,2,-3)||} = (-1,2,-3)\,\frac{1}{\sqrt{1+4+9}} = \left(\frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$.

EXERCISE 7.4. Prove the following identities:

(1) $||u+v||^2 + ||u-v||^2 = 2||u||^2 + 2||v||^2$.
(2) $u \cdot v = \frac{1}{4}||u+v||^2 - \frac{1}{4}||u-v||^2$.

Solution. Clearly, $||x||^2 = x \cdot x$. We will use this property to solve both questions.

(1) A direct computation shows

$$\begin{aligned} ||u+v||^2 + ||u-v||^2 &= (u+v)\cdot(u+v) + (u-v)\cdot(u-v) \\ &= u\cdot u + u\cdot v + v\cdot u + v\cdot v + u\cdot u - u\cdot v - v\cdot u + v\cdot v \\ &= 2\,(u\cdot u) + 2\,(v\cdot v) \\ &= 2||u||^2 + 2||v||^2. \end{aligned}$$

(2) Beginning with the right hand side this time,

$$\begin{aligned} \frac{1}{4}||u+v||^2 - \frac{1}{4}||u-v||^2 &= \frac{u\cdot u}{4} + \frac{u\cdot v}{4} + \frac{v\cdot u}{4} + \frac{v\cdot v}{4} - \left(\frac{u\cdot u}{4} - \frac{u\cdot v}{4} - \frac{v\cdot u}{4} + \frac{v\cdot v}{4}\right) \\ &= 2\frac{u\cdot v}{4} + 2\frac{v\cdot u}{4} \\ &= \frac{1}{2}u\cdot v + \frac{1}{2}u\cdot v \\ &= u\cdot v. \end{aligned}$$

EXERCISE 7.5. Fix $\mathbf{x} = (4,-3,6,2)$ and $\mathbf{y} = (6,1,7,7)$

(1) $2\mathbf{y} + 3\mathbf{x} =?$
(2) $\mathbf{x}\cdot\mathbf{y} =?$
(3) Is $\sqrt{\mathbf{x}\cdot\mathbf{x}} + \sqrt{\mathbf{y}\cdot\mathbf{y}} \geq \sqrt{(\mathbf{x}+\mathbf{y})\cdot(\mathbf{x}+\mathbf{y})}$?

Solution.

(1) $2\mathbf{y} + 3\mathbf{x} = (12,2,14,14) + (12,-9,18,6) = (24,-7,32,20)$.

(2) $\mathbf{x} \cdot \mathbf{y} = 24 - 3 + 42 + 14 = 77$.

(3) Let us compute both sides of the inequality

$$\begin{aligned}\sqrt{\mathbf{x} \cdot \mathbf{x}} + \sqrt{\mathbf{y} \cdot \mathbf{y}} &= \sqrt{16+9+36+4} + \sqrt{36+1+49+49} \\ &= \sqrt{65} + \sqrt{135}. \\ \sqrt{(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y})} &= \sqrt{(10,-2,13,9) \cdot (10,-2,13,9)} \\ &= \sqrt{100+4+169+81} = \sqrt{354},\end{aligned}$$

and the inequality holds.

Exercise 7.6. Fix $\mathbf{x} = (5, 0, -6, 2)$ and $\mathbf{y} = (3, 2, 3, 2)$

(1) $-4\mathbf{y} + 6\mathbf{x} =?$

(2) $\mathbf{x} \cdot \mathbf{y} =?$

(3) Is $\sqrt{\mathbf{x} \cdot \mathbf{x}} + \sqrt{\mathbf{y} \cdot \mathbf{y}} > \sqrt{(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y})}$?

Solution.

(1) $-4\mathbf{y} + 6\mathbf{x} = (-12, -8, -12, -8) + (30, 0, -36, 12) = (18, -8, -48, 4)$.

(2) $\mathbf{x} \cdot \mathbf{y} = 15 + 0 - 18 + 4 = 1$.

(3) Let us compute both sides of the inequality

$$\begin{aligned}\sqrt{\mathbf{x} \cdot \mathbf{x}} + \sqrt{\mathbf{y} \cdot \mathbf{y}} &= \sqrt{25+0+36+4} + \sqrt{9+4+9+4} \\ &= \sqrt{65} + \sqrt{26}. \\ \sqrt{(\mathbf{x}+\mathbf{y}) \cdot (\mathbf{x}+\mathbf{y})} &= \sqrt{(8,2,-3,4) \cdot (8,2,-3,4)} \\ &= \sqrt{64+4+9+16} = \sqrt{93},\end{aligned}$$

and the inequality holds.

7.2. Functions.

A function $f : A \to B$ is a rule that assign for every element $x \in A$, one and only one element in B, which is $f(x)$. When considering function $f : \mathbb{R}^n \to \mathbb{R}^m$ from $\mathbb{R}^n$ to $\mathbb{R}^n$, then we should understand that the input variable $x \in \mathbb{R}^n$ is a vector, and the output variable $f(x) \in \mathbb{R}^m$ is a vector as well.

The multivariate function could be as simple as the linear function $f(x, y) = x + y$, which is linear in each of its variables, and it could be a bit more complicated, such as $f(x, y, z) = (x + z, y - z)$.

Why are these functions so important? in elementary microeconomics, for example, we used a one-dimensional demand function $q = f(p)$ that simply depends on the price p. However, this model is quite limited. In general, the demand can depend on the price of the good p, as well as on the price of alternative goods p_a and the income y. Implying that $q = f(p, p_a, y) = c \cdot p^{c_1} p_a^{c_2} y^{c_3}$. In addition, if we take into account the demand of the alternative good, we get a mapping

$$q_T(p, p_a, y) = \left(c \cdot p^{c_1} p_a^{c_2} y^{c_3}, c' \cdot p^{c_4} p_a^{c_5} y^{c_6}\right).$$

Another example is production functions. For example, consider the following Cobb-Douglas production function[1]

$$q(K, L) = kK^{a_1} L^{a_2},$$

[1]The Cobb-Douglas production function is a production function, widely used to represent the technological relationship between the amounts of two or more inputs, particularly Capital and Labor, and the amount of output that can be produced by those inputs. The parameters a_1 and a_2 are the output elasticities of Capital and Labor, respectively. These values are constants determined by available technology. (Taken from Wikipedia).

and the Constant Elasticity of Substitutions (CES) production function[2]

$$q(K,L) = k\left(\lambda K^{-a} + (1-\lambda) L^{-a}\right)^{-1/a}.$$

Both relate to two means of production - capital K and labor L. Obviously, one can derive more general functions that depend on different elements and more variables.

EXAMPLE 7.1. A sports store in St. Louis carries two kinds of tennis rackets, the Serena Williams and the Maria Sharapova autograph brands. The consumer demand for each brand depends not only on its own price, but also on the price of the competing brand. Sales figures indicate that if the Williams brand sells for x dollars per racket and the Sharapova brand for y dollars per racket, the demand for Williams rackets will be $D_1(x,y) = 300 - 20x + 30y$ rackets per year and the demand for Sharapova rackets will be $D_2 = 200 + 40x - 10y$ rackets per year. Express the store's total annual revenue from the sale of these rackets as a function of the prices x and y.

Solution. Let R denote the total monthly revenue. Then

$$\begin{aligned} R(x,y) &= D_1(x,y)\,x + D_2(x,y)\,y \\ &= (300 - 20x + 30y)\,x + (200 + 40x - 10y)\,y \\ &= 300x + 200y + 70xy - 20x^2 - 10y^2. \end{aligned}$$

EXAMPLE 7.2. Output Q at a factory is often regarded as a function of the amount K of capital investment and the size L of the labor force. Suppose $Q(K,L) = \frac{3K^2+5L}{K-L}$. Find the domain of Q and compute $Q(2,1)$.

Solution. Since division by any real number except zero is possible, the expression $Q(K,L)$ can be evaluated for all ordered pairs (K,L) with $K - L \neq 0$ or $K \neq L$. Geometrically, this is the set of all points in the KL plane except for those on the line $K = L$.

$$Q(2,1) = \frac{3 \cdot (2)^2 + 5 \cdot (1)}{2-1} = 17.$$

EXAMPLE 7.3. Output Q at a factory is often regarded as a function of the amount K of capital investment and the size L of the labor force. Suppose $Q(K,L) = Ke^L + \ln(K)$. Find the domain of Q and compute $Q(e^2, \ln(2))$.

Solution. Since Ke^L is defined for all real numbers K and L and since $\ln(K)$ is defined only for $K > 0$, the domain of Q consists of all ordered pairs (K,L) of real numbers for which $K > 0$.

$$Q(e^2, \ln(2)) = e^2 e^{\ln(2)} + \ln\left(e^2\right) = 2e^2 + 2 \approx 16.78.$$

7.2.1. Graphs of functions.

When we move to higher dimensions, providing a geometric representation of a function is not so easy. Since we live in a 3dimensional world, comprehending how 4 dimensions look is problematic. Therefore, we are so what limited in that perspective. However, there are functions we can sketch, such as functions from $\mathbb{R}^2$ to $\mathbb{R}^1$ which are surfaces in $\mathbb{R}^3$. For example, the function $f(x,y) = x + y$ is linear in every coordinate, and its graph is a linear plane that goes through the points $(0,0,0)$ with a slope of 1 in the direction of the axis. Another example is $f(x,y) = x^2 + y^2$ which is a parabolic function in every coordinate, with a global minimum in $(0,0)$.

Another way to present a function graphically is by using level curves. Taking the function $f(x,y) = x^2 + y^2$, one can sketch the set $B_n = \left\{(x,y) \in \mathbb{R}^2 : \ x^2 + y^2 = n^2\right\}$ by drawing a circle in $\mathbb{R}^2$ with radius n for every $n \in \mathbb{N}$. In this case, we get a sketch of all the points on which the function equals n.

[2]Constant elasticity of substitution (CES), in economics, is a property of some production functions and utility functions. Specifically, it arises in a particular type of aggregation function which combines two or more types of consumption, or two or more types of productive inputs into an aggregate quantity. This aggregation function exhibits constant elasticity of substitution. (Taken from Wikipedia).

One of the simplest function we know is the linear function. A linear function also exist in higher (than one) dimensions. However, first we need to define a linear function. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if the followings properties hold for every two vectors $x, y \in \mathbb{R}^n$ and every real number $c \in \mathbb{R}$:

- $f(x+y) = f(x) + f(y)$.
- $f(cx) = cf(x)$.

THEOREM 7.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a linear function. then there exists an $m \times k$ matrix A such that $f(x) = Ax$ for every $x \in \mathbb{R}^n$.*

This theorem shows that a linear function could be represented by a matrix and thus, if we chose one coordinate and fix all others, we get a one dimensional linear function (straight line in one dimension).

7.3. Limits & continuity

When using the Euclidean norm, one can generalize the limit definition (Definition 2.1) directly.

DEFINITION 7.1. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a multivariate function and let $x_0 \in \mathbb{R}^n$ and $L \in \mathbb{R}^m$ be two real-valued vectors. L is the limit of f in the point x_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $\|x - x_0\| < \delta$, it follows that

$$\|f(x) - L\| < \epsilon.$$

We denote this limit by $\lim_{x \to x_0} f(x) = L$.
From the context, one can understand that $\|x - x_0\|$ is the distance between x and x_0 in $\mathbb{R}^n$, and $\|f(x) - L\|$ is the distance between L and $f(x)$ in $\mathbb{R}^m$. We use this definition to define continuous functions in the general case. A function f is continuous in x_0 if the following conditions holds:

(1) the function f has a limit L in x_0.
(2) the function f is defined in x_0.
(3) the equality $f(x_0) = L$ holds.

These are basically the same condition that we needed to ensure that a single-variable function is continuous. One type of functions that we will consider later is the class of continuous functions $f : \mathbb{R} \to \mathbb{R}^n$. These functions are called *curves*.
One way to compute limits in $\mathbb{R}^2$ is to substitute x, y with $r\cos(\theta), r\sin(\theta)$ when r is the distance of the point from the origin and θ is the angle the line between the point and the origin creates w.r.t the x axis. This way, we can replace the limit $(x, y) \to (0, 0)$ with the one dimensional limit $r \to 0$, which is easily computable.

EXERCISE 7.7. Find the limit the function $f(x, y) = \frac{xy}{x^2+y^2}$ in $(0, 0)$ along the curves:

(1) $y = 0$.
(2) $y(x) = x^2$.
(3) $y(x) = x$.

Solution. In every case we substitute y with the relevant $y(x)$ that defines the curve.

(1)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &\overset{y=0}{=} \lim_{x\to 0} \frac{x \cdot 0}{x^2 + 0^2} = \\ &= \lim_{x \to 0} 0 = 0.\end{aligned}$$

(2)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x, y) &\overset{y=x^2}{=} \lim_{x\to 0} \frac{x \cdot x^2}{x^2 + (x^2)^2} \\ &= \lim_{x\to 0} \frac{x^3}{x^2 + x^4} \\ &= \lim_{x\to 0} \frac{x}{1 + x^2} = 0.\end{aligned}$$

(3)

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &\overset{y=x}{=} \lim_{x\to 0}\frac{x\cdot x}{x^2+x^2}\\ &= \lim_{x\to 0}\frac{x^2}{2x^2}=\frac{1}{2}.\end{aligned}$$

we can see that, om general, the limit in $(0,0)$ does not exist, as different curves towards $(0,0)$ produce different outcomes.

EXERCISE 7.8. Compute the following limits. If the limit does not exist, prove it.

(1) $\lim_{(x,y)\to(0,0)}\frac{x^3+xy}{x^2+y^2}$.
(2) $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^4}$.
(3) $\lim_{(x,y)\to(0,0)} x\ln(|x|+|y|)$.

Solution.

(1) Let us compute the above limit a long the curves $y=0$ and $y=x$. If $y=0$ then

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^3+xy}{x^2+y^2} &= \lim_{x\to 0}\frac{x^3+x\cdot 0}{x^2+0^2}\\ &= \lim_{x\to 0}\frac{x^3}{x^2}\\ &= \lim_{x\to 0} x = 0.\end{aligned}$$

However, if $y=x$ then

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\frac{x^3+xy}{x^2+y^2} &= \lim_{x\to 0}\frac{x^3+x\cdot x}{x^2+x^2}\\ &= \lim_{x\to 0}\frac{x^3+x^2}{2x^2}\\ &= \lim_{x\to 0}\frac{x+1}{2}=\frac{1}{2},\end{aligned}$$

which means the the limit above does not exist.

(2) We compute the limit in absolute value.

$$\begin{aligned}\lim_{(x,y)\to(0,0)}\left|\frac{x^2y}{x^2+y^4}\right| &= \lim_{(x,y)\to(0,0)}\left|\frac{x^2}{x^2+y^4}\right|\cdot|y|\\ &= \lim_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^4}\cdot|y|\\ &\le \lim_{(x,y)\to(0,0)} 1\cdot|y| = 0.\end{aligned}$$

(3) Now, we use the polar coordinates $(r\cos(\theta), r\sin(\theta))$ to compute the limit.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} x\ln(|x|+|y|) =\\ &= \cos(\theta)\lim_{r\to 0^+} r\ln(|r\cos(\theta)|+|r\sin(\theta)|) =\\ &= \cos(\theta)\lim_{r\to 0^+} r\ln(r(|\cos(\theta)|+|\sin(\theta)|)) =\\ &= \cos(\theta)\lim_{r\to 0}\left[r\ln(r)+r\ln(|\cos(\theta)|+|\sin(\theta)|)\right].\end{aligned}$$

The term $r\ln(|\cos(\theta)|+|\sin(\theta)|)$ goes to zero as $r\to 0^+$. Therefore, we need to compute the limit of the first term.

$$\begin{aligned}\lim_{r\to 0^+} r\ln(r) &= \lim_{r\to 0^+}\frac{\ln(r)}{\frac{1}{r}} =\\ &\overset{"\frac{\infty}{\infty}"}{=} \lim_{r\to 0^+}\frac{\frac{1}{r}}{-\frac{1}{r^2}} =\\ &= \lim_{r\to 0^+} -r = 0.\end{aligned}$$

7.4. Partial derivatives

Although the definition of a continuous function is similar to the one-variable case, the same is not true when it comes to derivative. As the function depends on many variables, one can take a derivative with respect to each one differently. This leads us to the concept of *partial derivatives*. We start with the simple case of a function $f : \mathbb{R}^n \to \mathbb{R}^1$. The partial derivative with respect to x_i ($i = 1, \ldots, n$) in $\mathbf{x} = (x_1, \ldots, x_n)$ is

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h},$$

when $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, when the i^{th} coordinate is 1.
The partial derivative w.r.t. x_i in $\mathbf{x}$ tells us how the function changes with an infinitesimal change in $\mathbf{x}$. If we want to discuss the derivative of f we should understand that the function can act quite differently under infinitesimal changes in different coordinates. For example, $F(x, y) = x - y$ is increasing in x and decreasing in y. For this reason, the Gradient of $f : \mathbb{R}^n \to \mathbb{R}^1$ is a vector of n coordinates defined as follows:

$$DF(\mathbf{x}) = \left(\frac{\partial f(\mathbf{x})}{\partial x_1}, \ldots, \frac{\partial f(\mathbf{x})}{\partial x_n} \right).$$

We also denote it by $\nabla f(\mathbf{x})$. Every coordinate is a partial derivative of f.

7.4.1. First-order approximation in $\mathbb{R}^n$.

One possible use of this derivative is to approximate the value of the function near a known value. Namely, fix $\mathbf{x}^0 \in \mathbb{R}^n$ and consider the linear approximation (FOA) which follows

$$f(x) \approx f\left(\mathbf{x}^0\right) + \sum_{i=1}^{n} \frac{\partial f\left(\mathbf{x^0}\right)}{\partial x_i} \left(x_i - \mathbf{x}_i^0\right).$$

This representation of f using a linear approximation is identical to the approximation we studied in the one-dimensional case.
An alternative formulation is reached by taking $x = \mathbf{x}^0 + \Delta x$

$$f\left(\mathbf{x}^0 + \Delta x\right) = f\left(\mathbf{x}_1^0 + \Delta x_1, \ldots, \mathbf{x}_n^0 + \Delta x_n\right) \approx f\left(\mathbf{x^0}\right) + \sum_{i=1}^{n} \frac{\partial f\left(\mathbf{x}^0\right)}{\partial x_i} \Delta x_i.$$

In some cases, we use dx_i instead of Δx_i.

EXERCISE 7.9. It is estimated that the weekly output of a certain plant is given by the function $Q(x, y) = 1,200x + 500y + x^2y - x^3 - y^2$ units, where x is the number of skilled workers and y the number of unskilled workers employed at the plant. Currently the workforce consists of 30 skilled workers and 60 unskilled workers. Use marginal analysis to estimate the change in the weekly output that will result from the addition of 1 more skilled worker if the number of unskilled workers is not changed. Compare your result with the actual change.

Solution. The partial derivative

$$Q_x(x, y) = 1200 + 2xy - 3x^2$$

is the rate of change of output with respect to the number of skilled workers. For any values of x and y, this is an approximation of the number of additional units that will be produced each week if the number of skilled workers is increased from x to $x + 1$ while the number of unskilled workers is kept fixed at y. In particular, if the workforce is increased from 30 skilled and 60 unskilled workers to 31 skilled and 60 unskilled workers, the resulting change in output is approximately

$$Q(30, 60) = 1,200 + 2 \cdot 30 \cdot 60 - 3 \cdot 30^2 = 2,100.$$

Whereas, the actual change is

$$\begin{aligned} Q(31,60)-Q(30,60) &= 1,200x+500y+x^2y-x^3-y^2 \\ &= 91,469-89,400=2,069. \end{aligned}$$

EXERCISE 7.10. Consider the function $f(x,y)=3x^2y^2-9xy^3$.

(1) Find its partial derivatives using the definition.
(2) Find its partial derivatives using the rules of differentiation.

Solution. We need to find the partial derivatives w.r.t. x and w.r.t. y.

(1) We need to find the partial derivatives w.r.t. x and w.r.t. y.

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \lim_{h\to 0}\frac{3(x+h)^2y^2-9(x+h)y^3-3x^2y^2+9xy^3}{h} \\ &= \lim_{h\to 0}\frac{3x^2y^2+6xhy^2+3h^2y^2-9xy^3-9hy^3-3x^2y^2+9xy^3}{h} \\ &= \lim_{h\to 0}\frac{6xhy^2+3h^2y^2-9hy^3}{h} \\ &= \lim_{h\to 0}6xy^2+3hy^2-9y^3=6xy^2-9y^3. \end{aligned}$$

$$\begin{aligned} \frac{\partial f(x,y)}{\partial y} &= \lim_{h\to 0}\frac{3x^2(y+h)^2-9x(y+h)^3-3x^2y^2+9xy^3}{h} \\ &= \lim_{h\to 0}\frac{3x^2y^2+6x^2hy+3h^2x^2-9x\left(y^3+3y^2h+3yh^2+h^3\right)-3x^2y^2+9xy^3}{h} \\ &= \lim_{h\to 0}\frac{6x^2hy+3h^2x^2-9x\left(3y^2h+3yh^2+h^3\right)}{h} \\ &= \lim_{h\to 0}6x^2y+3hx^2-27xy^2-27xyh+-9xh^2=6x^2y-27xy^2. \end{aligned}$$

(2) Using the rules of differentiation yields

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \frac{\partial\left(3x^2y^2-9xy^3\right)}{\partial x} \\ &= 6xy^2-9y^3, \\ \frac{\partial f(x,y)}{\partial y} &= \frac{\partial\left(3x^2y^2-9xy^3\right)}{\partial y} \\ &= 6x^2y-27xy^2. \end{aligned}$$

EXERCISE 7.11. Consider the following Cobb-Douglas production function $q(K,L)=kK^{a_1}L^{a_2}$, and the Constant Elasticity of Substitutions (CES) production function $q(K,L)=k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-1/a}$. Compute the partial derivative of these functions assuming that all parameters are positive. Give an economic interpretation for these derivatives.

Solution. Let us begin with the Cobb-Douglas production function.

$$\begin{aligned} \frac{\partial q(K,L)}{\partial K} &= \frac{\partial\left(kK^{a_1}L^{a_2}\right)}{\partial K} \\ &= ka_1K^{a_1-1}L^{a_2}, \\ \frac{\partial q(K,L)}{\partial L} &= \frac{\partial\left(kK^{a_1}L^{a_2}\right)}{\partial L} \\ &= ka_2K^{a_1}L^{a_2-1}. \end{aligned}$$

We move on to the CES production function. Before we differentiate, we make a few small algebraic changes in the function.

$$\begin{aligned} k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-1/a} &= ke^{\ln\left[\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-1/a}\right]} \\ &= ke^{-\frac{\ln\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}{a}}. \end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial q(K,L)}{\partial K} &= \frac{\partial\left(ke^{-\frac{\ln\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}{a}}\right)}{\partial K} \\
&= ke^{-\frac{\ln\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}{a}}\left[-\frac{-a\lambda K^{-a-1}}{a\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}\right] \\
&= k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-1/a}\left[\frac{\lambda K^{-a-1}}{\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}\right] \\
&= k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-\frac{1}{a}-1}\lambda K^{-a-1}.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial q(K,L)}{\partial L} &= \frac{\partial\left(ke^{-\frac{\ln\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}{a}}\right)}{\partial L} \\
&= ke^{-\frac{\ln\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}{a}}\left[-\frac{-a(1-\lambda)L^{-a-1}}{a\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}\right] \\
&= k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-1/a}\left[\frac{(1-\lambda)L^{-a-1}}{\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)}\right] \\
&= k\left(\lambda K^{-a}+(1-\lambda)L^{-a}\right)^{-\frac{1}{a}-1}(1-\lambda)L^{-a-1}.
\end{aligned}$$

The derivative taken w.r.t. K presents the change in production given a small change in capital, while the derivative taken w.r.t. L presents the change in production given a small change in labor.

Exercise 7.12. Consider the production function $Q(K,L)=9K^{1/3}L^{2/3}$.

(1) What is the output when $K=216$ and $L=1000$?
(2) Use marginal analysis to estimate $Q(216,998)$ and $Q(217.5,1000)$. Compute these values up to three decimal places and compare with your estimation.

Solution.

(1) $Q(216,1000)=9(216)^{1/3}(1000)^{2/3}=9\cdot 6\cdot 100=5400$.
(2) We use the regular linear approximation,

$$Q(K,L+\Delta L) \approx Q(K,L)+\frac{\partial Q(K,L)}{\partial L}\Delta L,$$

and

$$Q(K+\Delta K,L) \approx Q(K,L)+\frac{\partial Q(K,L)}{\partial K}\Delta K.$$

We need to compute the partial derivatives w.r.t. K and L.

$$\frac{\partial Q(K,L)}{\partial K}=\frac{3L^{2/3}}{K^{2/3}}\ ;\ \frac{\partial Q(K,L)}{\partial L}=\frac{6K^{1/3}}{L^{1/3}}.$$

Hence,

$$\begin{aligned}
Q(216,998) &\approx Q(216,1000)+\frac{\partial Q(216,1000)}{\partial L}(-2) \\
&= 5400-\left[6\frac{6}{10}\right]\cdot 2 \\
&= 5400-7.2=5392.8,
\end{aligned}$$

while the true value is 5392.798. The error is -0.002.

$$\begin{aligned}
Q(217.5,1000) &\approx Q(216,1000)+\frac{\partial Q(216,1000)}{\partial K}(1.5) \\
&= 5400+\left[3\frac{100}{36}\right]\cdot\frac{3}{2} \\
&= 5400+12.5=5412.5,
\end{aligned}$$

while the true value is 5412.471. The error is -0.029.

EXERCISE 7.13. A firm has the Cobb-Douglas production function $Q = 10x_1^{1/3}x_2^{1/2}x_3^{1/6}$. It is currently using the input bundle $(27, 16, 64)$

(1) How much is it producing?
(2) Use differentials to approximate its new output when $x_1 = 27.1$, $x_2 = 15.7$, and x_3 remains the same.
(3) Compare your results for an exact computation with a calculator.

Solution.

(1) $Q(27, 16, 64) = 10 \cdot 27^{1/3} \cdot 16^{1/2} \cdot 64^{1/6} = 10 \cdot 3 \cdot 4 \cdot 2 = 240.$
(2) The relevant partial derivatives are

$$\frac{\partial Q(x_1, x_2, x_3)}{\partial x_1} = \frac{10x_2^{1/2}x_3^{1/6}}{3x_1^{2/3}}, \quad \frac{\partial Q(x_1, x_2, x_3)}{\partial x_2} = \frac{5x_1^{1/3}x_3^{1/6}}{x_2^{1/2}}.$$

Thus,

$$\begin{aligned} Q(27.1, 15.7, 64) &= Q(27, 16, 64) + \frac{\partial Q(27, 16, 64)}{\partial x_1}\Delta x_1 + \frac{\partial Q(27, 16, 64)}{\partial x_2}\Delta x_2 \\ &= 240 + 0.1 \cdot \frac{10 \cdot 16^{1/2} \cdot 64^{1/6}}{3 \cdot 27^{2/3}} - 0.3 \cdot \frac{5 \cdot 27^{1/3} \cdot 64^{1/6}}{16^{1/2}} \\ &= 240 + 0.1 \cdot \frac{10 \cdot 4 \cdot 2}{3 \cdot 9} - 0.3 \cdot \frac{5 \cdot 3 \cdot 2}{4} = 238.046. \end{aligned}$$

(3) The actual output is 238.032, an error of 0.14.

7.4.2. The Jacobian matrix and The Hessian matrix.

To generalize this notion, let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. That is, $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$ when f_i is a function from $\mathbb{R}^n$ to $\mathbb{R}^1$. We can apply the same notion once again to every f_i and get that the derivative, usually called the *Jacobian* (or, *Jacobian matrix* | *derivative*) is a matrix

$$DF(x^0) = \begin{pmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \frac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_1(\mathbf{x})}{\partial x_n} \\ \frac{\partial f_2(\mathbf{x})}{\partial x_1} & \frac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m(\mathbf{x})}{\partial x_1} & \frac{\partial f_m(\mathbf{x})}{\partial x_2} & \cdots & \frac{\partial f_m(\mathbf{x})}{\partial x_n} \end{pmatrix}.$$

The Jacobian is also denoted by $J_f(\mathbf{x})$.
The *Hessian*, or *Hessian matrix,* is similar to the Jacobian, where the derivatives are of the second order instead of the first order. The Hessian of a function $f : \mathbb{R}^n \to \mathbb{R}$ is

$$D^2F(\mathbf{x}) = H_F(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_1} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_n \partial x_1} \\ \frac{\partial f^2(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial f^2(\mathbf{x})}{\partial x_2^2} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f^2(\mathbf{x})}{\partial x_1 \partial x_n} & \frac{\partial f^2(\mathbf{x})}{\partial x_2 \partial x_n} & \cdots & \frac{\partial f^2(\mathbf{x})}{\partial x_n^2} \end{pmatrix}.$$

It is easy to generalize this to the general case of a function $f : \mathbb{R}^n \to \mathbb{R}^m$, as every coordinate of f, i.e., every f_i, is differentiated w.r.t. every x_i and x_j.
When high-order derivatives are concerned, there is one important theorem that helps us with the computation, and that is Young's theorem.

THEOREM 7.2. **(Young's theorem)** *Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ is a C^2 function, then for every coordinates i, j, k it holds that*

$$\frac{\partial^2 f_k}{\partial x_i \partial x_j} = \frac{\partial^2 f_k}{\partial x_j \partial x_i}.$$

Young's theorem tells us that the order by which we differentiate is not important when the function are at least twice continuously differential.

EXERCISE 7.14. Using Young's theorem, compute the third order partial derivatives of $Q = 4K^{3/4}L^{1/4}$.

Solution. We compute the partial derivatives up to the third order explicitly.

$$\begin{aligned} \frac{\partial Q}{\partial K} &= 3K^{-\frac{1}{4}}L^{\frac{1}{4}} \quad , \quad \frac{\partial Q}{\partial L} = K^{\frac{3}{4}}L^{-\frac{3}{4}}, \\ \frac{\partial^2 Q}{\partial K^2} &= -\frac{3}{4}K^{-\frac{5}{4}}L^{\frac{1}{4}} \quad , \quad \frac{\partial^2 Q}{\partial L^2} = -\frac{3}{4}K^{\frac{3}{4}}L^{-\frac{7}{4}}, \\ \frac{\partial^2 Q}{\partial L \partial K} &= \tfrac{\partial^2 Q}{\partial K \partial L} = \frac{3}{4}K^{-\frac{1}{4}}L^{-\frac{3}{4}}. \end{aligned}$$

And the third order derivatives are

$$\begin{aligned} \frac{\partial^3 Q}{\partial K^3} &= \frac{15}{16}K^{-\frac{9}{4}}L^{\frac{1}{4}} \quad , \quad \frac{\partial^3 Q}{\partial L^3} = \frac{21}{16}K^{\frac{3}{4}}L^{-\frac{11}{4}}, \\ \frac{\partial^3 Q}{\partial L \partial K^2} &= -\frac{3}{16}K^{-\frac{5}{4}}L^{-\frac{3}{4}} \quad , \quad \frac{\partial^3 Q}{\partial K \partial L^2} = -\frac{9}{16}K^{-\frac{1}{4}}L^{-\frac{7}{4}}, \end{aligned}$$

and all the other derivative are given by Young's theorem.

EXERCISE 7.15. In this question we examine a function such that Young's theorem does not hold.

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) = (0,0), \\ \frac{x^3y - xy^3}{x^2+y^2}, & \text{if } (x,y) \neq (0,0). \end{cases}$$

(1) Prove that the partial derivatives in $(0,0)$ are both zero.
(2) Compute the partial derivative for any point (x,y).
(3) Compute the partial derivative in $(0,y)$ and in $(x,0)$.
(4) Prove that the second partial derivatives in $(0,0)$ are not equal.

Solution.

(1) When $x = 0$, then $f(0,y) = \frac{0^3y - 0y^3}{0^2+y^2} = 0$ and the same holds for $f(x,0)$, thus

$$\begin{aligned} \frac{\partial f(0,0)}{\partial x} &= \lim_{x\to 0}\frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0, \\ \frac{\partial f(0,0)}{\partial y} &= \lim_{y\to 0}\frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0. \end{aligned}$$

(2) We can use the rules of differentiation and get

$$\begin{aligned} \frac{\partial f(x,y)}{\partial x} &= \frac{\left(3x^2y - y^3\right)\left(x^2+y^2\right) - 2x\left(x^3y - xy^3\right)}{\left(x^2+y^2\right)^2} \\ &= \frac{3x^4y + 3x^2y^3 - y^3x^2 - y^5 - 2x^4y + 2x^2y^3}{\left(x^2+y^2\right)^2} \\ &= \frac{x^4y + 4x^2y^3 - y^5 - x^3y^2}{\left(x^2+y^2\right)^2}. \end{aligned}$$

$$\begin{aligned} \frac{\partial f(x,y)}{\partial y} &= \frac{\left(x^3 - 3xy^2\right)\left(x^2+y^2\right) - 2y\left(x^3y - xy^3\right)}{\left(x^2+y^2\right)^2} \\ &= \frac{x^5 + x^3y^2 - 3x^3y^2 - 3xy^4 - 2x^3y^2 + 2xy^4}{\left(x^2+y^2\right)^2} \\ &= \frac{x^5 - 4x^3y^2 - xy^4}{\left(x^2+y^2\right)^2}. \end{aligned}$$

(3) Using the previous computation yields

$$\begin{aligned} \frac{\partial f(0,y)}{\partial y} &= 0 \quad , \quad \frac{\partial f(x,0)}{\partial y} = x, \\ \frac{\partial f(0,y)}{\partial x} &= -y \quad , \quad \frac{\partial f(x,0)}{\partial x} = 0. \end{aligned}$$

(4) We compute the second derivatives using the definition.

$$\begin{aligned}\frac{\partial^2 f(0,0)}{\partial x \partial y} &= \lim_{x\to 0}\frac{\frac{\partial f(x,0)}{\partial y}-\frac{\partial f(0,0)}{\partial y}}{x} = \lim_{x\to 0}\frac{x-0}{x} = 1,\\ \frac{\partial^2 f(0,0)}{\partial y \partial x} &= \lim_{y\to 0}\frac{\frac{\partial f(0,y)}{\partial x}-\frac{\partial f(0,0)}{\partial x}}{y} = \lim_{x\to 0}\frac{-y-0}{y} = -1.\end{aligned}$$

We can see that the second derivative in $(0,0)$ are not equal.

EXERCISE 7.16. Consider the production function $Q = K^{3/4}L^{3/4}$. Show that the marginal productivity of each factor is diminishing. Show, however, that if the input combination is doubled, then output more than doubles.

Solution. We need to compute the partial derivatives.

$$\frac{\partial Q}{\partial K} = \frac{3L^{3/4}}{4K^{1/4}}, \ \frac{\partial Q}{\partial L} = \frac{3K^{3/4}}{4L^{1/4}}.$$

We can see that the productivity diminishes in every factor. However,

$$\begin{aligned}Q(2K,2L) &= 2^{6/4}K^{3/4}L^{3/4}\\ &= 2^{1.5}K^{3/4}L^{3/4} > 2Q(K,L).\end{aligned}$$

7.4.3. The chain rule.

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *continuously differential* on a set U if the partial derivative $\frac{\partial f_j(\mathbf{x})}{\partial x_i}$ in every coordinates i, j and in every point $\mathbf{x} \in U$ exists, and it is continuous in $\mathbf{x}$. In other words, a function is continuously differential if it has all its partial derivatives and all of them are continuous. The set of functions that are continuously differential is denoted by C^1. Moreover, the set of functions that are n times continuously differential is denoted by C^n.

When composing two functions, the derivative of the composition is based on the derivative of both functions we used. For that purpose we have the chain rule. In the one-dimensional case, the chain rule was relatively simple. However, the generalization is a bit more complicated. Thus, we make this generalization in two stages. First we present the chain rule for curves and later on for general functions.

THEOREM 7.3. **(The chain rule for curves)** *Assume that* $\mathbf{x}(t) = (x_1(t), \dots, x_n(t))$ *is a* C^1 *curve*[3] *and* $f : \mathbb{R}^n \to \mathbb{R}$ *is also* C^1. *Then* $g(t) = (f \circ \mathbf{x})(t) = f(x_1(t), \dots, x_n(t))$ *is also* C^1 *and*

$$\begin{aligned}g'(t) = \frac{dg(t)}{dt} &= \frac{\partial f(\mathbf{x}(t))}{\partial x_1}\cdot\frac{dx_1(t)}{dt} + \cdots + \frac{\partial f(\mathbf{x}(t))}{\partial x_n}\cdot\frac{dx_n(t)}{dt}\\ &= \frac{\partial f(\mathbf{x}(t))}{\partial x_1}x_1'(t) + \cdots + \frac{\partial f(\mathbf{x}(t))}{\partial x_n}x_n'(t) =\\ &= \nabla f(\mathbf{x}(t)) \cdot (x_1'(t), \dots, x_n'(t)).\end{aligned}$$

We see that the derivatives of the composition is the inner product of the gradients of both functions.

THEOREM 7.4. **(The chain rule for general functions)** *Let* $f : \mathbb{R}^n \to \mathbb{R}^m$ *and* $g : \mathbb{R}^k \to \mathbb{R}^n$ *be two* C^1 *functions such that the composition* $h = f \circ g : \mathbb{R}^k \to \mathbb{R}^m$ *is well defined. Then* $h(\mathbf{x}) = (f \circ g)(\mathbf{x})$ *is also* C^1 *and the Jacobian of* h *is given by*

$$\begin{aligned}Dh(\mathbf{x}) &= Df(g(\mathbf{x})) \cdot Dg(\mathbf{x}) =\\ J_h(\mathbf{x}) &= J_f(g(\mathbf{x})) \cdot J_g(\mathbf{x}).\end{aligned}$$

In words, the Jacobian of h *is the matrix product of the Jacobian of* f *in* $g(\mathbf{x})$ *and the Jacobian of* g *in* $\mathbf{x}$.

[3] As we do not want to discuss the domain and co-domain of every function, we assume that the composition of functions is well defined.

Note that the Jacobian of f is an $m \times n$ matrix and the Jacobian of g is an $n \times k$ matrix, which means that the matrix product of both is well defined and yields an $m \times k$ matrix, as the Jacobian of h should be.

EXERCISE 7.17. At a given moment in time, the marginal product of labor is 2.5 and the marginal product of capital is 3, the amount of capital is increasing by 2 each unit of time and the rate of change of labor is +0.5. What is the rate of change of output w.r.t. time?

Solution. The rate of change of output is $\Delta Q = 2.5 \cdot 0.5 + 3 \cdot 2 = 7.25$.

EXERCISE 7.18. Let $f(x, y) = 3xy^2 + 2x$ where $x(t) = -3t^2$ and $y(t) = 4t^3 + t$.

(1) Use the chain rule to find how $f(x(t), y(t))$ changes as a function of t.
(2) Use substitution and direct differentiation to compute how f changes as a function of t.

Solution.

(1) With the chain rule we get

$$\begin{aligned}\frac{d}{dt} f(x(t), y(t)) &= \frac{\partial f}{\partial x} \cdot x'(t) + \frac{\partial f}{\partial y} \cdot y'(t) \\ &= \left(3y^2(t) + 2\right)(-6t) + 6x(t)\, y(t) \cdot \left(12t^2 + 1\right).\end{aligned}$$

(2) Using direct differentiation we get

$$\begin{aligned}\frac{d}{dt} f(x(t), y(t)) &= \frac{d}{dt} f\left(-3t^2, 4t^3 + t\right) \\ &= \frac{d}{dt}\left[3\left(-3t^2\right)\left(4t^3 + t\right)^2 + 2\left(-3t^2\right)\right] \\ &= \frac{d}{dt}\left[-9t^2\left(4t^3 + t\right)^2 - 6t^2\right] \\ &= -3\frac{d}{dt}\left[t^2\left(3\left(4t^3 + t\right)^2 + 2\right)\right] \\ &= -6t\left(4t^3 + t\right)\left[3\left(4t^3 + t\right) + 2 + 3t\left(12t^2 + 1\right)\right].\end{aligned}$$

EXERCISE 7.19. A health store carries two kinds of vitamin water, brand A and brand B. Sales figures indicate that if brand A is sold for x dollars per bottle and brand B for y dollars per bottle, the demand for brand A will be $Q(x, y) = 300 - 20x^2 + 30y$ bottles per month. It is estimated that t months from now the price of brand A will be $x = 2 + 0.05t$ dollars per bottle, and the price of brand B will be $y = 2 + 0.1\sqrt{t}$ dollars per bottle. At what rate will the demand for brand A be changing with respect to time 4 months from now?

Solution. Your goal is to find $\frac{dQ}{dt}$ when $t = 4$. Using the chain rule, you get

$$\frac{dQ}{dt} = \frac{\partial q}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial q}{\partial y} \cdot \frac{dy}{dt} = -40x \cdot (0.05) + 30 \cdot \left(0.05t^{-0.5}\right).$$

When $t = 4$, we get $x = 2 + 0.05 \cdot 4 = 2.2$, hence

$$\frac{dQ}{dt} = -40 \cdot 2.2 \cdot 0.05 + 30 \cdot 0.05 \cdot 0.5 = -3.65.$$

That is, 4 months from now the monthly demand for brand A will be decreasing at the rate of 3.65 bottles per month.

EXERCISE 7.20. Let $w(r, s)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that $r = y - x$ and that $s = y + x$. Define $F(x, y) = w(r(x, y), s(x, y))$. Compute the partial derivatives of F in terms of the partial derivatives of w.

Solution. A direct computation shows

$$\begin{aligned}
\frac{\partial F}{\partial x} &= \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x}+\frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x}\\
&= -\frac{\partial w}{\partial r}+\frac{\partial w}{\partial s};\\
\frac{\partial F}{\partial y} &= \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial y}+\frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial y}\\
&= \frac{\partial w}{\partial r}+\frac{\partial w}{\partial s}.
\end{aligned}$$

7.5. The Implicit function theorem

The functions we dealt with so far were *explicit functions*. For example, $y = f(x_1, x_2, \ldots, x_n)$ as the endogenous variable y is a function of the exogenous variables $x_1, \ldots, x_n$. There are functions, commonly known as *implicit functions*, where the variables cannot be separated as in the previous case, so that $F(x_1, x_2, \ldots, x_n, y) = 0$ represents y as an implicit function of the variables $x_1, \ldots, x_n$. For example, $y^3 - 3xy + x^2 - 7 = 0$. For every value of x, we can solve the equation for y and get a value (when there is more than one value we can choose one). This implies that y is a function of x, but this function cannot be represented generally in the form $y = f(x)$, because x and y cannot be algebraically separated.
When considering these functions, we wish to know whether we can represent y explicitly as a function of the other variables, and also compute it's derivative. For that we have the *Implicit Function Theorem.*

THEOREM 7.5. **(Implicit Function Theorem)** *Let $F(x, y)$ be a C^1 function where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Assume there exists a vector $(x^0, y_0) = (x_1^0, \ldots, x_n^0, y_0)$ such that*

$$\begin{aligned}
F(x^0, y_0) &= Constant;\\
\frac{\partial F(x^0, y_0)}{\partial y} &\neq 0.
\end{aligned}$$

Then, there exists a C^1 function $Y = Y(x_1, \ldots, x_n)$ such that

(1) $F(x, Y(x)) = c$, for every vector $x \in \mathbb{R}^n$ close to x^0;
(2) $Y(x^0) = y_0$, and
(3) $\frac{\partial Y(x^0)}{\partial x_i} = -\frac{\frac{\partial F(x^0, y_0)}{\partial x_i}}{\frac{\partial F(x^0, y_0)}{\partial y}}$, for every coordinate x_i.

The motivation behind the theorem is to construct a function Y which, essentially, represents the variable y, at least locally, around x^0. We do not know how the function Y looks like, so we do not have an explicit formulations for it. However, we do know the value of the function at x^0,which is y_0,we do know that the function sustain the equality w.r.t. F (where $F(x, Y(x)) = c$), and most importantly, we know the derivative of Y w.r.t. every x_i. Thus, by the first and third point above, we can produce a first-order approximation of Y w.r.t. every other coordinate, and this first-order approximation will give us some intuition of the behavior of the variable y, according to the x_is, at least locally around x^0.
For example, consider the previous example $F(x, y) = y^3 - 3xy + x^2 = 7$ and the point $(x_0, y^0) = (4, 3)$. We can see that the point sustains the condition $F(4, 3) = 7$. If we compute that partial derivatives of F we get,

$$\begin{aligned}
\frac{\partial F(x, y)}{\partial x} = -3y + 2x \quad &, \quad \frac{\partial F(x, y)}{\partial y} = 3y^2 - 3x,\\
\frac{\partial F(4, 3)}{\partial x} = -1 \quad &, \quad \frac{\partial F(4, 3)}{\partial y} = 15.
\end{aligned}$$

Thus, the conditions of the Implicit Function Theorem, Theorem 7.5, hold, so we know that a C^1 function $Y(x)$ exists around $x = 4$, that sustains the three conditions above. That is,

(1) $F(x, Y(x)) = 7$, for every x near 4,

(2) $Y(4) = 3$, and
(3) $Y'(4) = -\frac{\frac{\partial F(4,3)}{\partial x}}{\frac{\partial F(4,3)}{\partial y}} = \frac{1}{15}$.

Why does that help us? Well, assume that we want to know the value of y when $x = 4.3$. Then we can use the first-order approximation and get

$$\begin{aligned} Y(4.3) &\approx Y(4) + y'(4)\,\Delta x \\ &= 3 + \frac{1}{15}\cdot 0.3 = 3.02. \end{aligned}$$

The true value in this case is $F(4.3, y) = y^3 - 12.9y + 18.49 = 7$. Solving the equation yields

$$y^3 - 12.9y + 11.49 = 0 \Rightarrow y = 3.01475.$$

EXERCISE 7.21. Prove the the expression $x^2 - xy^3 + y^5 = 17$ is an implicit function of y in terms of x around the point $(x^0, y^0) = (5, 2)$. Estimate the value of y when $x = 4.8$.

Solution. First, define $F(x, y) = x^2 - xy^3 + y^5$. We can see that $5^2 - 5\cdot 2^3 + 2^5 = 25 - 40 + 32 = 17$, as needed. In addition,

$$\frac{\partial F(x,y)}{\partial x} = 2x - y^3 \quad , \quad \frac{\partial F(x,y)}{\partial y} = -3xy^2 + 5y^4,$$
$$\frac{\partial F(5,2)}{\partial x} = 2 \quad , \quad \frac{\partial F(5,2)}{\partial y} = -20,$$

and the second condition of Theorem 7.5 holds as well. The estimation yields

$$\begin{aligned} y(4.8) &= y(5) + y'(5)\,\Delta x \\ &= 2 + \left(-\frac{\frac{\partial FG(5,2)}{\partial x}}{\frac{\partial F(5,2)}{\partial y}}\right)(-0.2) = \\ &= 2 - \frac{2}{20}\cdot\frac{1}{5} = 1.98. \end{aligned}$$

EXERCISE 7.22. Consider the function $F(x_1, x_2, y) = x_1^2 - x_2^2 + y^3$.

(1) If $x_1 = 6$ and $x_2 = 3$ then find a y which satisfies $F(x_1, x_2, y) = 0$.
(2) Does the equation define y as a function of (x_1, x_2) near $(6, 3)$?
(3) If so, compute the partial derivative of y in $(6, 3)$.
(4) If $(x_1, x_2) = (6.2, 2.9)$ estimate the value of y.

Solution.

(1) We need to solve the equation

$$\begin{aligned} 36 - 9 + y^3 &= 0, \\ y^3 &= -27, \\ y &= -3. \end{aligned}$$

(2) We need to see whether the second condition of the Implicit Function Theorem holds.

$$\begin{aligned} \frac{\partial F(x_1, x_2, y)}{\partial y} &= 3y^2, \\ \frac{\partial F(6, 3, -3)}{\partial y} &= 27 \neq 0. \end{aligned}$$

Thus, the conditions hold.

(3) Lets us compute the partial derivatives of the function F,

$$\frac{\partial F(x_1, x_2, y)}{\partial x_1} = 2x_1 \quad , \quad \frac{\partial F(x_1, x_2, y)}{\partial x_2} = -2x_2,$$
$$\frac{\partial F(6, 3, -3)}{\partial x_1} = 12 \quad , \quad \frac{\partial F(6, 3, -3)}{\partial x_2} = -6.$$

Therefore,

$$\begin{aligned}
\frac{\partial y(6,3)}{\partial x_1} &= -\frac{\frac{\partial F(6,3,-3)}{\partial x_1}}{\frac{\partial F(6,3,-3)}{\partial y}} \\
&= -\frac{12}{27} = -\frac{4}{9}, \\
\frac{\partial y(6,3)}{\partial x_2} &= -\frac{\frac{\partial F(6,3,-3)}{\partial x_2}}{\frac{\partial F(6,3,-3)}{\partial y}} \\
&= -\frac{-6}{27} = \frac{2}{9}.
\end{aligned}$$

(4) We can use first-order approximation and get

$$\begin{aligned}
y(6.2, 2.9) &= y(6,3) + \frac{\partial y(6,3)}{\partial x_1}\Delta x_1 + \frac{\partial y(6,3)}{\partial x_2}\Delta x_2 \\
&= -3 - \frac{4}{9}\cdot\frac{2}{10} - \frac{2}{9}\cdot\frac{1}{10} = -3.111.
\end{aligned}$$

EXERCISE 7.23. Consider the function $3x^2yz + xyz^2 = 30$ as defining x as an implicit function of (y, x) around $(1, 3, 2)$.

(1) Estimate x when $(y, z) = (3.2, 2)$.
(2) Solve the equation $3x^2yz+xyz^2 = 30$ explicitly to find x as a function of y, z. Use approximation to estimate x when $(y, z) = (3.2, 2)$. Which way was easier?

Solution.

(1) We need to find the derivative of x w.r.t. y.

$$\begin{aligned}
\frac{\partial x(y,z)}{\partial y} &= -\frac{\frac{\partial\left(3x^2yz+xyz^2\right)}{\partial y}}{\frac{\partial(3x^2yz+xyz^2)}{\partial x}} \\
&= -\frac{3x^2z + xz^2}{6xyz + yz^2} \\
&= -\frac{5}{24}, \text{ at } (1,3,2).
\end{aligned}$$

Hence,

$$\begin{aligned}
x(3.2, 2) &= x(3,2) + \frac{\partial x(y,z)}{\partial y}\Delta y \\
&= 1 - \frac{5}{24}\cdot 0.2 = \frac{23}{24}.
\end{aligned}$$

(2) Solving the second-order equation $3x^2yz + xyz^2 - 30 = 0$ yields

$$\begin{aligned}
x &= \frac{-yz^2 + \sqrt{y^2z^4 + 360yz}}{6yz} \\
\frac{\partial x(y,z)}{\partial y} &= \frac{6yz\left(-z^2 + \frac{2yz^2+360z}{2(y^2z^4+360yz)}\right) - \left(-yz^2 + \sqrt{y^2z^4+360yz}\right)6z}{36y^2z^2}.
\end{aligned}$$

When plugging in $(3, 2)$ we get $-\frac{5}{24}$, as before. Clearly the first method was much easier.

EXERCISE 7.24. Suppose the output at a certain factory is $Q(x, y) = 2x^3 + x^2y + y^3$ units, where x is the number of hours of skilled labor used and y is the number of hours of unskilled labor. The current labor force consists of 30 hours of skilled labor and 20 hours of unskilled labor. Estimate the change in unskilled labor y that should be made to offset a 1-hour increase in skilled labor x, so that output will be maintained at its current level.

Solution. The current level of output is the value of Q when $x = 30$ and $y = 20$. That is,

$$Q(30, 20) = 2\cdot 30^3 + 30^2\cdot 20 + 20^3 = 80,000.$$

If output is to be maintained at this level, the relationship between skilled labor x and unskilled labor y is given by the equation

$$80,000 = 2x^3 + x^2 y + y^3,$$

which defines y implicitly as a function of x. The goal is to estimate the change in y that corresponds to a 1-unit increase in x when x and y are related by this equation. The change in y caused by a 1-unit increase in x can be approximated by the derivative $\frac{dy}{dx}$. To find this derivative, we use implicit differentiation.

$$\begin{aligned} 0 &= 6x^2 + x^2\frac{dy}{dx} + 2xy + 3y^2\frac{dy}{dx} \\ \frac{dy}{dx}\left(-x^2 - 3y^2\right) &= 6x^2 + 2xy \\ \frac{dy}{dx} &= -\frac{6x^2 + 2xy}{x^3 + 3y^2} = -\frac{6 \cdot 30^2 + 2 \cdot 30 \cdot 20}{30^3 + 3 \cdot 20^2} = -3.14. \end{aligned}$$

That is, to maintain the current level of output, unskilled labor should be decreased by approximately 3.14 hours to offset a 1-hour increase in skilled labor.

7.6. Multidimensional Integrals

7.6.1. Integrals in several variables.

Integrating a function with several variables is not that different from one-variable integration. Similarly to differentiation, when integrating several variables we relate to the other variables as constant and integrate the function as if it was a one-variable function.

However, there are times that we wish to compute one integral before the other. That is, assume that $f(x, y)$ is an integrable function (which means that we can compute its integral which is finite), and assume that we need to compute

$$\int_0^1 \int_0^y f(x, y)\, dxdy.$$

There are cases, where it is easier to first integrate the y variable and only later the x variable. For these cases, we have *Fubini's Theorem.*

THEOREM 7.6. **(Fubini's Theorem)** *If the function $f(x, y)$ is integrable and the integral is finite, then*

$$\int\int f(x, y)\, dxdy = \int \left[\int f(x, y)\, dx\right] dy = \int \left[\int f(x, y)\, dy\right] dx.$$

7.6.2. Differentiating integrals.

Often we want to differentiate an objective function to find an optimum, and when the objective function has an integral we need to know how to differentiate it. There is a rule for doing so, called *Leibniz's rule*, named after the 17th-century German mathematician who was one of the two independent inventors of calculus (along with Newton). We want to find

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x, t)dx.$$

Note that we are differentiating with respect to t, and we are integrating with respect to x. Nevertheless, t shows up three times in the expression, once in the upper limit of the integral, $b(t)$, once in the lower limit of the integral, $a(t)$, and once in the integrand, $f(x, t)$. We need to figure out what to do with these three terms.

Three things happen when t changes. First, the function $f(x, t)$ shifts. Second, the right endpoint $b(t)$ changes, and third, the left endpoint $a(t)$ changes. Leibniz's rule accounts for all three of these shifts. Leibniz's rule says

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x, t)dx = \int_{a(t)}^{b(t)} \frac{\partial f(x, t)}{\partial t}dx + b'(t)f(b(t), t) - a'(t)f(a(t), t).$$

Each of the three terms corresponds to one of the shifts we mentioned. The first term accounts for the shift of the curve $f(x,t)$. The term $\frac{\partial f(x,t)}{\partial t}$ tells how far the curve shifts at point x, and the integral $\int_{a(t)}^{b(t)} \frac{\partial f(x,t)}{\partial t} dx$ tells how much the area changes because of the shift in $f(x,t)$. The second term accounts for the movement in the right endpoint, $b(t)$. The third term accounts for the movement in the left endpoint, $a(t)$. Putting these three terms together gives us Leibniz's rule, which looks complicated but hopefully makes sense.

One of the important uses for this operation is in the field of auction theory. In auctions, when searching for an equilibrium, we sometimes need to optimize function that are based on integrals. We will discuss such examples broadly when we study probability.

EXERCISE 7.25. Compute the following derivative. First by using Leibniz's rule, and then by integrating and taking the required derivative.

(1) $\frac{d}{dt}\int_{-t^2}^{t^2} tx^2 dx$.
(2) $\frac{d}{dt}\int_{-3t}^{4t^2} t^2x^3 dx$.

Solution. We start by differentiating both integrals directly.

$$\begin{aligned}
\frac{d}{dt}\int_{-t^2}^{t^2} tx^2 dx &= \int_{-t^2}^{t^2} 1\cdot x^2 dx + 2t\cdot\left(t\cdot\left(t^2\right)^2\right) - (-2t)\cdot\left(t\cdot\left(-t^2\right)^2\right)\\
&= \int_{-t^2}^{t^2} x^2 dx + 2t^6 + 2t^6\\
&= \frac{t^6}{3} - \frac{-t^6}{3} + 4t^6\\
&= \frac{14}{3}t^6.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_{-3t}^{4t^2} t^2x^3 dx &= \int_{-3t}^{4t^2} 2tx^3 dx + 8t\cdot\left(t^2\cdot\left(4t^2\right)^3\right) - (-3)\cdot\left(t^2\cdot(-3t)^3\right)\\
&= 2t\int_{-3t}^{4t^2} x^3 dx + 2^9t^9 - 3^4t^5\\
&= t\left[\frac{\left(4t^2\right)^4}{2} - \frac{(-3t)^4}{2}\right] + 2^9t^9 - 3^4t^5\\
&= 2^7t^9 - \frac{3^4}{2}t^5 + 2^9t^9 - 3^4t^5\\
&= 5\cdot 2^7t^9 - \frac{3^5}{2}t^5.
\end{aligned}$$

Now, we first integrate and then take the derivative.

$$\begin{aligned}
\frac{d}{dt}\int_{-t^2}^{t^2} tx^2 dx &= \frac{d}{dt}\left[\frac{t\left(t^2\right)^3}{3} - \frac{t\left(-t^2\right)^3}{3}\right]\\
&= \frac{d}{dt}\left[\frac{2t^7}{3}\right]\\
&= \frac{14}{3}t^6.
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}\int_{-3t}^{4t^2} t^2x^3dx &= \frac{d}{dt}\left[t^2\frac{\left(4t^2\right)^4}{4} - t^2\frac{\left(-3t\right)^4}{4}\right] \\
&= \frac{d}{dt}\left[4^3t^{10} - \frac{3^4}{4}t^6\right] \\
&= 10\cdot 2^6t^9 - 6\cdot\frac{3^4}{4}t^5 \\
&= 5\cdot 2^7t^9 - \frac{3^5}{2}t^5.
\end{aligned}$$

CHAPTER 8

Optimization

8.1. Unconstrained optimization

Though functions of more than one variable are complicated, finding an extreme point of such a function is quite similar to the one-dimensional case. Similarly to Fermat's Theorem, Theorem 2.3, when a function $f:\mathbb{R}^n \to \mathbb{R}^1$ is C^1 and it has a local minimum (or, a local maximum) $\mathbf{x}$, then $\frac{\partial f(\mathbf{x})}{\partial x_i} = 0$ for every $i = 1,\ldots,n$.

THEOREM 8.1. *Let $f:\mathbb{R}^n \to \mathbb{R}^1$ be a C^1 function with an interior local minimum (or, a local maximum) $\mathbf{x}$, then $\frac{\partial f(\mathbf{x})}{\partial x_i} = 0$ for every $i = 1,\ldots,n$.*

Again, the fact that the partial derivatives are zero, does not imply that the point is an extreme point. It is the other way around.
One can use the Hessian of f to determined whether the point is a minimum, a maximum, or a saddle point.

LEMMA 8.1. *Let $f:\mathbb{R}^n \to \mathbb{R}^1$ be a C^1 function where all the partial derivative in a point $\mathbf{x}$ are zero (we assume that $\mathbf{x}$ is an inner point of the domain of f).*

- If the Hessian $H_f(\mathbf{x})$ is a negative (semi-) definite symmetric matrix, then f is a strictly (weakly) concave function, which implies that $\mathbf{x}$ is a strict (weak) local maximum of f.
- If the Hessian $H_f(\mathbf{x})$ is a positive (semi-) definite symmetric matrix, then f is a strictly (weakly) convex function, which implies that $\mathbf{x}$ is a strict (weak) local minimum of f.
- If the Hessian $H_f(\mathbf{x})$ is a indefinite symmetric matrix, then f is neither concave nor convex, thus $\mathbf{x}$ is neither a local minimum of f, nor a local maximum of f.

In case you do not remember what the definiteness of a matrix is, you should go over Section 6.2 once more. Here is a quick reminder of Theorem 6.1.

LEMMA 8.2. *Let $f:\mathbb{R}^n \to \mathbb{R}^1$ be a C^1 function where all the partial derivative in a point $\mathbf{x}$ are zero (we assume that $\mathbf{x}$ is an inner point of the domain of f).*

- The Hessian $H_f(\mathbf{x})$ is positive definite (semi-positive definite) if all principle minor are strictly positive (non-negative, respectively) . That is, if

$$\left|\frac{\partial^2 f}{\partial x_1^2}\right| > 0, \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} > 0, \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix} > 0, \ldots$$

  Alternatively, the Hessian is positive definite if all its eigenvalues are positive. (Semi-positive definite follows from either non-negative eigenvalues or non-negative principle minors.)
- The Hessian $H_f(\mathbf{x})$ is negative definite (semi-negative definite) if have alternating signs: the minors of odd order are strictly positive and the others are strictly negative. That is, if

$$\left|\frac{\partial^2 f}{\partial x_1^2}\right| > 0, \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} < 0, \quad \begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_3 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_3^2} \end{vmatrix} > 0, \ldots$$

Alternatively, the Hessian is negative definite if all its eigenvalues are negative. (Semi-negative definite follows from either non-positive eigenvalues or alternating signs in principle minors with weak inequalities.)

- If the principle minors of $H_f(\mathbf{x})$ violate the two previous patterns, then $\mathbf{x}$ is a saddle point.

EXERCISE 8.1. The production function of a firm is $F(K,L) = K^{1/4}L^{1/2}$. The retail price of the product is 12, and the unit price of capital and labor are 6 each. Find the optimal values for K and L.

Solution. The profit function is

$$\pi(K,L) = 12K^{1/4}L^{1/2} - 6K - 6L.$$

The first order conditions are

$$\begin{aligned} \frac{\partial \pi(K,L)}{\partial K} &= 3K^{-3/4}L^{1/2} - 6 = 0, \\ \frac{\partial \pi(K,L)}{\partial L} &= 6K^{1/4}L^{-1/2} - 6 = 0. \end{aligned}$$

Thus,

$$\begin{aligned} \frac{L^{1/2}}{K^{3/4}} = 2 \quad &\Rightarrow \quad L^2 = 16K^3, \\ \frac{K^{1/4}}{L^{1/2}} = 1 \quad &\Rightarrow \quad K = L^2. \end{aligned}$$

And we get $K = 0.25, L = 0.5$. The Hessian is

$$\begin{aligned} \begin{pmatrix} \frac{\partial^2 f}{\partial K^2} & \frac{\partial^2 f}{\partial L \partial K} \\ \frac{\partial^2 f}{\partial K \partial L} & \frac{\partial^2 f}{\partial L^2} \end{pmatrix} &= \begin{pmatrix} -\frac{9}{4}K^{-7/4}L^{1/2} & \frac{3}{2}K^{-3/4}L^{-1/2} \\ \frac{3}{2}K^{-3/4}L^{-1/2} & -3K^{1/4}L^{-3/2} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{9}{4}\cdot 0.25^{-7/4}0.5^{1/2} & \frac{3}{2}\cdot 0.25^{-3/4}0.5^{-1/2} \\ \frac{3}{2}\cdot 0.25^{-3/4}0.5^{-1/2} & -3\cdot 0.25^{1/4}0.5^{-3/2} \end{pmatrix}. \end{aligned}$$

But computing the leading principal minors we get that the point is a strict local maximum.

EXERCISE 8.2. A monopolist is facing two distinct markets - a domestic market and a foreign one. Let Q_i be the amount supplied to market i, and let $P_i = G_i(Q_i)$ be the inverse demand function of market i. Specifically, the revenue from market i is $Q_iP_i = Q_iG_i(Q_i)$ when

$$G_1(Q_1) = 50 - Q_1, \; G_2(Q_2) = 100 - 10Q_2.$$

The cost function of the firm is

$$C(Q_1 + Q_2) = C(Q) = 90 + 20Q.$$

Find how much should the monopoly produce for each market in order to maximize profit.

Solution. The monopolist profit function is

$$\begin{aligned} \pi(Q_1,Q_2) &= Q_1(50 - Q_1) + Q_2(100 - Q_2) - 90 - 20(Q_1 + Q_2) = \\ &= 30Q_1 - Q_1^2 + 80Q_2 - Q_2^2 - 90. \end{aligned}$$

The FOC conditions show that

$$\begin{aligned} \frac{\partial \pi(Q_1,Q_2)}{\partial Q_1} &= 30 - 2Q_1 = 0, \\ \frac{\partial \pi(Q_1,Q_2)}{\partial Q_2} &= 80 - 2Q_2 = 0. \end{aligned}$$

Thus,

$$Q_1 = 15, \; Q_2 = 40.$$

For the SOC we get

$$\begin{pmatrix} \frac{\partial^2 \pi(Q_1,Q_2)}{\partial Q_1^2} & \frac{\partial^2 \pi(Q_1,Q_2)}{\partial Q_2 \partial Q_1} \\ \frac{\partial^2 \pi(Q_1,Q_2)}{\partial Q_1 \partial Q_2} & \frac{\partial^2 \pi(Q_1,Q_2)}{\partial Q_2^2} \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

One can see that the first LPM is -2 and the second LPM is 4, which means that the bundle is profit maximizing.

EXERCISE 8.3. Find the critical points of the functions

$$\begin{aligned} f(x,y) &= 4x^2 + 3y^2 - 12xy + 18x. \\ f(x,y) &= 16xy - 4x + 2y^{-1}. \end{aligned}$$

Solution. Taking the FOC for $f(x,y) = 4x^2 + 3y^2 - 12xy + 18x$ yields

$$\begin{aligned} f_x(x,y) &= 8x - 12y + 18 = 0, \\ f_y(x,y) &= 6y - 12x = 0. \end{aligned}$$

Thus, $y = 2x$ and

$$\begin{aligned} 8x - 24x + 18 &= 0 \\ &\Downarrow \\ 16x &= 18 \\ x &= \frac{9}{8}, y = \frac{9}{4}. \end{aligned}$$

Taking the FOC for $f(x,y) = 16xy - 4x + 2y^{-1}$ yields

$$\begin{aligned} f_x(x,y) &= 16y - 4 = 0, \\ f_y(x,y) &= 16x - 2y^{-2} = 0. \end{aligned}$$

And $y = \frac{1}{4}$,

$$x = \frac{1}{8y^2} = 2.$$

EXERCISE 8.4. A firm has a Cobb-Douglas production function

$$Q(x,y) = x^a y^b.$$

It faces output prices of p, and input prices of w, r respectively. Find the profit-maximizing input bundle. Find condition on the parameters such that this solution is a global maximum.

Solution. The profit function is

$$\pi(x,y) = px^a y^b - wx - ry.$$

The FOC yield

$$\begin{aligned} \pi_x(x,y) &= apx^{a-1}y^b - w = 0, \\ \pi_y(x,y) &= bpx^a y^{b-1} - r = 0. \end{aligned}$$

Solving this system gives $y = \frac{bw}{ar}x$, and so

$$\begin{aligned} \frac{w}{ap} &= x^{a-1}\left(\frac{bw}{ar}x\right)^b \\ x &= \left[\frac{w^{1-b}r^b}{a^{1-b}pb^b}\right]^{1/(a+b-1)}, \\ y &= \frac{bw}{ar}\left[\frac{w^{1-b}r^b}{a^{1-b}pb^b}\right]^{1/(a+b-1)}. \end{aligned}$$

The Hessian is

$$\begin{pmatrix} a(a-1)px^{a-2}y^b & abpx^{a-1}y^{b-1} \\ abpx^{a-1}y^{b-1} & b(b-1)px^a y^{b-2} \end{pmatrix}.$$

Thus the first LPM is $a(a-1)px^{a-2}y^b < 0$ if and only if $a \in (0,1)$. The second LPM is

$$\begin{aligned} ab\left[(a-1)(b-1)p^2x^{2a-2}y^{2b-2} - abp^2x^{2a-2}y^{2b-2}\right] &= abp^2x^{2a-2}y^{2b-2}\left[(a-1)(b-1)-ab\right] \\ &= abp^2x^{2a-2}y^{2b-2}\left[1-a-b\right]. \end{aligned}$$

And if $a+b<1,\ a \in (0,1)$, and $b \in (0,1)$, this is a global maximum.

8.2. Optimization with constraints

8.2.1. Equality constraints.

In the previous section we studied unconstrained optimization problems. We had a function $f : \mathbb{R}^n \to \mathbb{R}^1$ and we needed to maximize or minimize it. However, many problems in economics have constraints. For example, consider a simple consumers problem. If the consumers do not have a budget constraint, and given that more consumption is better for the consumers, a consumer would choose an infinite amount of every good. Clearly this cannot be achieved in the real world, because consumers cannot purchase unlimited quantities of goods. The budget constraint is another condition that we need to take into account when trying to optimize the utility of the consumer. How do we that? Well, it turns out to be not that difficult due a very simple, yet ingenious, function called *The Lagrangian.*

8.2.1.1. *Lagrangian.*

The Lagrangian is named after the person who developed it, a 18$^{\text{th}}$ century Italian-French mathematician, called Joseph-Louis Lagrange. The Lagrangian takes the function we wish to optimize and the conditions we need to sustain and bundles into one function that we need to optimize. It is a very elegant and easy way to solve constrained optimization problems. How does it work?

(1) Assume that we wish to maximize a function $f(\mathbf{x})$ when $f : \mathbb{R}^n \to \mathbb{R}^1$, but we have the following m equality constraints given by the equations

$$\begin{aligned} g_1(\mathbf{x}) &= a_1, \\ g_2(\mathbf{x}) &= a_2, \\ &\vdots \\ g_m(\mathbf{x}) &= a_m, \end{aligned}$$

where $g_i : \mathbb{R}^n \to \mathbb{R}^1$ is a C^1 function and $a_i \in \mathbb{R}$, for every $i = 1,\dots,m$.

(2) First, we write down the Lagrangian $L : \mathbb{R}^{n+m} \to \mathbb{R}^1$,

$$L(\mathbf{x}, \lambda_1, \dots, \lambda_m) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \left[a_i - g_i(\mathbf{x})\right].$$

Note that L is a function of $\mathbf{x}$ and of $\lambda_1, \dots, \lambda_m$, where λ_i are called the *Lagrange multipliers.*

(3) Next we use the FOC (first-order condition) on L. That is, we compare the partial derivatives to 0 and find the critical points,

$$\begin{aligned} \frac{\partial L(\mathbf{x}, \lambda_1, \dots, \lambda_m)}{\partial x_1} &= 0, \quad \dots \quad , \frac{\partial L(\mathbf{x}, \lambda_1, \dots, \lambda_m)}{\partial x_n} = 0, \\ \frac{\partial L(\mathbf{x}, \lambda_1, \dots, \lambda_m)}{\partial \lambda_1} &= 0, \quad \dots \quad , \frac{\partial L(\mathbf{x}, \lambda_1, \dots, \lambda_m)}{\partial \lambda_n} = 0. \end{aligned}$$

The following theorem concludes this procedure.

THEOREM 8.2. *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ and $g_i : \mathbb{R}^n \to \mathbb{R}^1$ where$i = 1, \ldots, m$ be C^1 functions, and consider the problem of maximizing (or minimizing) $f(\mathbf{x})$ given the constraints*

$$\begin{aligned} g_1(\mathbf{x}) &= a_1, \\ g_2(\mathbf{x}) &= a_2, \\ &\vdots \\ g_m(\mathbf{x}) &= a_m, \end{aligned}$$

where $a_i \in \mathbb{R}$ for every $i = 1, \ldots, m$. If $\mathbf{x}*$ is a solution for this problem then there exists $(\lambda_1^*, \ldots, \lambda_m^*)$ such that

$$\frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial x_1} = 0, \quad \ldots \quad , \frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial x_n} = 0,$$
$$\frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial \lambda_1} = 0, \quad \ldots \quad , \frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial \lambda_n} = 0.$$

In other words, the theorem states that if a solution exists, then we should find it by the FOC of The Lagrangian.
The best way to understand this procedure at this point is by exercising it several times, thus we move along to solving problems.

EXERCISE 8.5. A consumer has a utility function

$$u(x_1, x_2) = x_1^{0.5} x_2^{0.5},$$

The prices are $p_1 = 10$, $p_2 = 20$ and his budget is $M = 120$. Find the bundle that maximizes his utility.

Solution. The maximizing problem is

$$\begin{aligned} \max_{x_1, x_2} u(x_1, x_2) = & \quad x_1^{0.5} x_2^{0.5}, \\ \text{s.t.} \quad 10x_1 + 20x_2 = & \quad 120. \end{aligned}$$

The Lagrangian is

$$L(x_1, x_2, \lambda) = x_1^{0.5} x_2^{0.5} + \lambda(120 - 10x_1 - 20x_2).$$

The FOCs give

$$\begin{aligned} L_{x_1}(x_1, x_2, \lambda) &= \frac{\sqrt{x_2}}{2\sqrt{x_1}} - 10\lambda = 0, \\ L_{x_2}(x_1, x_2, \lambda) &= \frac{\sqrt{x_1}}{2\sqrt{x_2}} - 20\lambda = 0, \\ L_{\lambda}(x_1, x_2, \lambda) &= 120 - 10x_1 - 20x_2 = 0. \end{aligned}$$

Solving this system yields

$$x_1 = 6, \ x_2 = 3, \ \lambda = \frac{1}{20\sqrt{2}}.$$

EXERCISE 8.6. Solve the maximization problem

$$\begin{aligned} \max_{x_1, x_2} f(x_1, x_2) = & \quad 3x_1x_2 + 4x_1, \\ \text{s.t.} \quad 4x_1 + 12x_2 = & \quad 80. \end{aligned}$$

Solution. The Lagrangian is

$$L(x_1, x_2, \lambda) = 3x_1x_2 + 4x_1 + \lambda(80 - 4x_1 - 12x_2).$$

The FOCs give

$$\begin{aligned}L_{x_1}(x_1,x_2,\lambda) &= 3x_2+4-4\lambda=0,\\ L_{x_2}(x_1,x_2,\lambda) &= 3x_1-12\lambda=0,\\ L_{\lambda}(x_1,x_2,\lambda) &= 80-4x_1-12x_2=0.\end{aligned}$$

Solving this system yields

$$x_1=12,\ x_2=\frac{8}{3},\ \lambda=3.$$

Exercise 8.7. Solve the minimization problem

$$\begin{aligned}\min_{x_1,x_2} f(x_1,x_2) = &\quad 5x_1+2x_2,\\ \text{s.t.}\qquad 3x_1+2x_1x_2 = &\quad 80.\end{aligned}$$

Solution. The Lagrangian is

$$L(x_1,x_2,\lambda)=5x_1+2x_2+\lambda(80-3x_1-2x_1x_2).$$

The FOCs give

$$\begin{aligned}L_{x_1}(x_1,x_2,\lambda) &= 5-3\lambda-2\lambda x_2=0,\\ L_{x_2}(x_1,x_2,\lambda) &= 2-2\lambda x_1=0,\\ L_{\lambda}(x_1,x_2,\lambda) &= 80-3x_1-2x_1x_2=0.\end{aligned}$$

Solving this system yields two results and the one that minimizies the function is

$$x_1=-4,\ x_2=-11\frac{1}{2},\ \lambda=-\frac{1}{4}.$$

Exercise 8.8. Solve the maximization problem

$$\begin{aligned}\max_{x_1,x_2} f(x_1,x_2) = &\quad x_1x_2,\\ \text{s.t.}\qquad x_1+4x_2 = &\quad 16.\end{aligned}$$

Solution. The Lagrangian is

$$L(x_1,x_2,\lambda)=x_1x_2+\lambda(16-x_1-4x_2).$$

The FOCs give

$$\begin{aligned}L_{x_1}(x_1,x_2,\lambda) &= x_2-\lambda=0,\\ L_{x_2}(x_1,x_2,\lambda) &= x_1-4\lambda=0,\\ L_{\lambda}(x_1,x_2,\lambda) &= 16-x_1-4x_2=0.\end{aligned}$$

Solving this system yields

$$x_1=8,\ x_2=2,\ \lambda=2.$$

Exercise 8.9. Solve the maximization problem

$$\begin{aligned}\max_{x_1,x_2} f(x_1,x_2) = &\quad x_1^2x_2,\\ \text{s.t.}\qquad 2x_1^2+x_2^2 = &\quad 3.\end{aligned}$$

Solution. The Lagrangian is

$$L(x_1,x_2,\lambda)=x_1^2x_2+\lambda\left(3-2x_1^2-x_2^2\right).$$

The FOCs give

$$\begin{aligned} L_{x_1}(x_1,x_2,\lambda) &= 2x_1x_2 - 4\lambda x_1 = 0, \\ L_{x_2}(x_1,x_2,\lambda) &= x_1^2 - 2\lambda x_2 = 0, \\ L_{\lambda}(x_1,x_2,\lambda) &= 3 - 2x_1^2 - x_2^2 = 0. \end{aligned}$$

Solving this system yields several solutions, thus we can plug each solution into the goal function and see which is the maximizer. The final solution is

$$x_1 = 1 \quad , \quad x_2 = 1,$$

or

$$x_1 = -1 \quad , \quad x_2 = 1.$$

EXERCISE 8.10. Solve the maximization problem

$$\begin{aligned} & \max_{x_1,x_2,x_3} f(x_1,x_2,x_3) = && x_1x_2x_3, \\ \text{s.t.} \quad & x_1^2 + x_2^2 = && 1, \\ & x_1 + x_3 = && 1. \end{aligned}$$

Solution. The Lagrangian is

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2) = x_1x_2x_3 + \lambda_1\left(1 - x_1^2 - x_2^2\right) + \lambda_2(1 - x_1 - x_3).$$

The FOCs give

$$\begin{aligned} L_{x_1}(x_1,x_2,x_3,\lambda_1,\lambda_2) &= x_2x_3 - 2\lambda_1x_1 - \lambda_2 = 0, \\ L_{x_2}(x_1,x_2,x_3,\lambda_1,\lambda_2) &= x_1x_3 - 2\lambda_1x_2 = 0, \\ L_{x_3}(x_1,x_2,x_3,\lambda_1,\lambda_2) &= x_1x_2 - \lambda_2 = 0, \\ L_{\lambda_1}(x_1,x_2,x_3,\lambda_1,\lambda_2) &= 1 - x_1^2 - x_2^2 = 0, \\ L_{\lambda_2}(x_1,x_2,x_3,\lambda_1,\lambda_2) &= 1 - x_1 - x_3 = 0. \end{aligned}$$

The final solution is

$$x_1 \approx -0.7676, \quad x_2 = -0.6409, \quad x_3 = 1.7676.$$

EXERCISE 8.11. Find the optimal bundle for the general Cobb-Douglas utility function

$$u(x_1,x_2) = kx_1^a x_2^{1-a},$$

on the budget set $p_1x_1 + p_2x_2 = I$.

Solution. The Lagrangian is

$$L(x_1,x_2,\lambda) = kx_1^a x_2^{1-a} + \lambda(I - p_1x_1 - p_2x_2).$$

The FOCs give

$$\begin{aligned} L_{x_1}(x_1,x_2,\lambda) &= kax_1^{a-1}x_2^{1-a} - \lambda p_1 = 0, \\ L_{x_2}(x_1,x_2,\lambda) &= k(1-a)x_1^a x_2^{-a} - \lambda p_2 = 0, \\ L_{\lambda}(x_1,x_2,\lambda) &= I - p_1x_1 - p_2x_2 = 0. \end{aligned}$$

Solving this system yields

$$x_1 = \frac{aI}{p_1}, \; x_2 = \frac{(1-a)I}{p_2}.$$

EXERCISE 8.12. Find the point closest to the origin that is on both planes

$$\begin{aligned} 3x + y + z &= 5, \\ x + y + z &= 1. \end{aligned}$$

Solution. We need to solve the minimization problem

$$\begin{aligned} \min_{x,y,z} f(x,y,z) = \quad & x^2+y^2+z^2, \\ \text{s.t.} \qquad 3x+y+z = \quad & 5, \\ x+y+z = \quad & 1. \end{aligned}$$

The Lagrangian is

$$L(x,y,z,\lambda_1,\lambda_2) = x^2+y^2+z^2+\lambda_1(5-3x-y-z)+\lambda_2(1-x-y-z).$$

The FOCs give

$$\begin{aligned} L_x(x,y,z,\lambda_1,\lambda_2) &= 2x-3\lambda_1-\lambda_2 = 0, \\ L_y(x,y,z,\lambda_1,\lambda_2) &= 2y-\lambda_1-\lambda_2 = 0, \\ L_z(x,y,z,\lambda_1,\lambda_2) &= 2z-\lambda_1-\lambda_2 = 0, \\ L_{\lambda_1}(x,y,z,\lambda_1,\lambda_2) &= 5-3x-y-z = 0, \\ L_{\lambda_2}(x,y,z,\lambda_1,\lambda_2) &= 1-x-y-z = 0. \end{aligned}$$

The solution we get from this system is

$$(x,y,z) = \left(2,-\frac{1}{2},-\frac{1}{2}\right).$$

8.2.2. Inequality constraints.

The previous problems focused on equality constraints. But what happens when some of the constraints are in the form of inequalities? For example, instead of the constraints set

$$\begin{aligned} g_1(\mathbf{x}) &= a_1, \\ g_2(\mathbf{x}) &= a_2, \\ &\vdots \\ g_m(\mathbf{x}) &= a_m, \end{aligned}$$

we have

$$\begin{aligned} g_1(\mathbf{x}) &= a_1, \\ g_2(\mathbf{x}) &= a_2, \\ &\vdots \\ g_k(\mathbf{x}) &= a_k, \\ g_{k+1}(\mathbf{x}) &\leq a_{k+1}, \\ &\vdots \\ g_m(\mathbf{x}) &\leq a_m. \end{aligned}$$

What happens now? The setup is quite simple to the one we had before. We still form the Lagrangian, but now we take a few different conditions as the following theorem states.

Remark 8.1. We now focus on maximization problems. When we have inequality constraints, then distinction is important, thus the following formulation holds for the maximization problem depicted.

THEOREM 8.3. *Let $f : \mathbb{R}^n \to \mathbb{R}^1$ and $g_i : \mathbb{R}^n \to \mathbb{R}^1$ where$i = 1, \ldots, m$ be C^1 functions, and consider the problem of maximizing $f(\mathbf{x})$ given the constraints*

$$\begin{aligned} g_1(\mathbf{x}) &= a_1, \\ g_2(\mathbf{x}) &= a_2, \\ &\vdots \\ g_k(\mathbf{x}) &= a_k, \\ g_{k+1}(\mathbf{x}) &\leq a_{k+1}, \\ &\vdots \\ g_m(\mathbf{x}) &\leq a_m. \end{aligned}$$

where $a_i \in \mathbb{R}$ for every $i = 1, \ldots, m$. Define the Lagrangian function

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \left[a_i - g_i(\mathbf{x})\right].$$

If $\mathbf{x}*$ is a solution for this problem, then there exists $(\lambda_1^*, \ldots, \lambda_m^*)$ such that

$$\frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial x_i} = 0, \quad \text{for every} \quad i = 1, \ldots, n,$$

$$\frac{\partial L(\mathbf{x}*, \lambda_1^*, \ldots, \lambda_m^*)}{\partial \lambda_i} = 0, \quad \text{for every} \quad i = 1, \ldots, k,$$

and for every inequality constraint $i = k+1, \ldots, m$, it holds that

$$\begin{cases} \lambda_i^*(a_i - g_i(\mathbf{x}*)) = 0, \\ \lambda_i^* \geq 0, \\ g_i(\mathbf{x}*) \leq a_i. \end{cases}$$

Note that the formulation of the Lagrangian is such that all constraints are taken as non-negative constraints. That is, we write $a_i - g_i(\mathbf{x}) \geq 0$ which is non negative, instead of $g_i(\mathbf{x}) - a_i \leq$ which is non positive.

REMARK 8.2. What happens if we have a minimization problem? In that case, the formulation is similar to the one presented in Theorem 8.3, but the Lagrangian is formulated with a minus sign, such that

$$L(\mathbf{x}, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) - \sum_{i=1}^{m} \lambda_i \left[a_i - g_i(\mathbf{x})\right].$$

8.2.2.1. *Kuhn-Taker Formulation.*

When the constraints are of the form

$$\begin{aligned} x_1 &\geq 0, \\ x_2 &\geq 0, \\ &\vdots \\ x_n &\geq 0, \\ g_{n+1}(\mathbf{x}) &\leq a_{n+1}, \\ &\vdots \\ g_m(\mathbf{x}) &\leq a_m, \end{aligned}$$

that is, when there are $m+n$ inequality constraints, when the first n relate to the non negativity of the coordinates of the solution, the Lagrangian of the maximization is defined as

$$L(\mathbf{x},\lambda_1,\ldots,\lambda_m)=f(\mathbf{x})+\sum_{i=1}^{n}\lambda_i x_i+\sum_{i=n+1}^{m}\lambda_i\left[a_i-g_i(\mathbf{x})\right], \tag{8.2.1}$$

which is identical to the Lagrangian presented in Theorem 8.3 with the relevant constraints. Yet, as this setup is quite common in economic problems, the Lagrangian presented in Equation 8.2.1 is called the *Kuhn-Tucker Lagrangian*. It is named after its developers, Harold Kuhn and A.W. Tucker.

Exercise 8.13. Solve the maximization problem

$$\begin{array}{lrl} & \max_{x_1,x_2} f(x_1,x_2)= & x_1-x_2^2,\\ \text{s.t.} & x_1^2+x_2^2= & 4,\\ & x_1\geq & 0,\\ & x_2\geq & 0.\end{array}$$

Solution. The Lagrangian is

$$L(x_1,x_2,\lambda_1,\lambda_2,\lambda_3)=x_1-x_2^2+\lambda_1x_1+\lambda_2x_2+\lambda_3\left(4-x_1^2-x_2^2\right).$$

The conditions we need to sustain are

$$\begin{array}{rcl} L_{x_1}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3) & = & 1+\lambda_1-2\lambda_3x_1=0,\\ L_{x_2}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3) & = & -2x_2+\lambda_2-2\lambda_3x_2=0,\\ L_{\lambda_3}(x_1,x_2,\lambda_1,\lambda_2,\lambda_3) & = & 4-x_1^2-x_2^2=0,\\ \lambda_1x_1=0, & & \lambda_2x_2=0,\\ \lambda_i\geq 0, & \text{for} & i=1,2,\\ x_i\geq 0, & \text{for} & i=1,2.\end{array}$$

We can see that $1+\lambda_1=2\lambda_3x_1$ implies that $x_1>0$ and $\lambda_3>0$ (since $\lambda_1\geq 0$). Thus, the equation $\lambda_1x_1=0$ implies that $\lambda_1=0$. Take the second equation are write it down as $2x_2(1+\lambda_3)=\lambda_2$. Since $1+\lambda_3>0$, we can deduce that either λ_2 and x_2 are both strictly positive or both are zero. By $\lambda_2x_2=0$, we conclude that $\lambda_2=x_2=0$. Thus,

$$\begin{array}{rl} x_1^2+0=4 & \Rightarrow\ x_1=2,\\ & \Rightarrow\ \lambda_1=0,\\ & \Rightarrow\ \lambda_3=\dfrac{1}{4}.\end{array}$$

The final solution is

$$(x_1,x_2,\lambda_1,\lambda_2,\lambda_3)=\left(2,0,0,0,\frac{1}{4}\right).$$

Exercise 8.14. Solve the maximization problem

$$\begin{array}{lrl} & \max_{x_1,x_2} f(x_1,x_2,x_3)= & x_1x_2x_3,\\ \text{s.t.} & x_1+x_2+x_3\leq & 1,\\ & x_1\geq & 0,\\ & x_2\geq & 0,\\ & x_3\geq & 0.\end{array}$$

Solution. The Lagrangian is

$$L(x_1,x_2,x_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4)=x_1x_2x_3+\lambda_1x_1+\lambda_2x_2+\lambda_3x_3+\lambda_4(1-x_1-x_2-x_3).$$

The conditions we need to sustain are

$$\begin{aligned}
L_{x_1}(x_1,x_2,x_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4) &\quad=\quad& x_2x_3+\lambda_1-\lambda_4=0,\\
L_{x_2}(x_1,x_2,x_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4) &\quad=\quad& x_1x_3+\lambda_2-\lambda_4=0,\\
L_{x_3}(x_1,x_2,x_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4) &\quad=\quad& x_1x_2+\lambda_3-\lambda_4=0,\\
\lambda_4(1-x_1-x_2-x_3)=0, &&\lambda_1x_1=0,\\
\lambda_2x_2=0, &&\lambda_3x_3=0,\\
\lambda_i\geq 0, &\quad\text{for}\quad& i=1,2,3,4,\\
x_i\geq 0, &\quad\text{for}\quad& i=1,2,3,\\
&\text{and,}\quad& x_1+x_2+x_3\leq 1.
\end{aligned}$$

The first three equations can be written as

$$\lambda_4=x_2x_3+\lambda_1=x_1x_3+\lambda_2=x_1x_2+\lambda_3.$$

We need to separate the problem into two cases: $\lambda_4=0$ or $\lambda_4>0$. If $\lambda_4=0$, by the non negativity of the variables we get that $\lambda_i=0\ \forall i=1,2,3,4$, and

$$x_2x_3=x_1x_2=x_1x_3=0.$$

Thus, the solution is that two variables equal zero, and the last one equals any number in $[0,1]$. Specifically, the objective function equals zero, and clearly this is a minimum point and not a maximum given the above-mentioned conditions. Now assume that $\lambda_4>0$. Thus, we get

$$1-x_1-x_2-x_3=0,$$

and at least one coordinate is strictly positive. Assume that $x_1=0$. Thus,the equations

$$\lambda_4=x_2x_3+\lambda_1=x_1x_3+\lambda_2=x_1x_2+\lambda_3,$$

and the fact $\lambda_4>0$ yield

$$\lambda_2=\lambda_3=\lambda_4>0.$$

This means that $x_2=x_3=0$ which contradicts the conclusion that $x_1+x_2+x_3=1$. Thus $x_1>0$, and by symmetry, the same holds for x_2 and x_3. Hence, $\lambda_1=\lambda_2=\lambda_3=0$, and

$$\begin{gathered}
x_2x_3=\quad x_1x_3\quad =x_1x_2\\
\Downarrow\\
x_2=\quad x_1=\quad x_3,
\end{gathered}$$

and the solution is

$$(x_1,x_2,x_3,\lambda_1,\lambda_2,\lambda_3,\lambda_4)\quad=\quad\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0,\frac{1}{9}\right).$$

Exercise 8.15. Solve the minimization problem

$$\begin{aligned}
&\min_{x_1,x_2} f(x_1,x_2)= && -x_1^2+2x_2,\\
\text{s.t.}\qquad & x_1^2+x_2^2\leq && 1,\\
& x_1\geq && 0,\\
& x_2\geq && 0.
\end{aligned}$$

Solution. Note that this is a minimization problem and so the Lagrangian is

$$L(x_1,x_2,\lambda_1,\lambda_2,\lambda_3)=-x_1^2+2x_2-\lambda_1x_1-\lambda_2x_2-\lambda_3\left(1-x_1^2-x_2^2\right).$$

The conditions we need to sustain are

$$\begin{aligned} L_{x_1}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) &= -2x_1 - \lambda_1 + 2\lambda_3 x_1 = 0, \\ L_{x_2}(x_1, x_2, \lambda_1, \lambda_2, \lambda_3) &= 2 - \lambda_2 + 2\lambda_3 x_2 = 0, \\ \lambda_1 x_1 = 0, & \qquad \lambda_2 x_2 = 0, \\ \lambda_3\left(1 - x_1^2 - x_2^2\right) = 0 & \\ \lambda_i \geq 0, & \quad \text{for} \quad i = 1, 2, 3 \\ x_i \geq 0, & \quad \text{for} \quad i = 1, 2, \\ & \quad \text{and,} \quad x_1^2 + x_2^2 \leq 1. \end{aligned}$$

Writing the equation $2 + 2\lambda_3 x_2 = \lambda_2$ and using the non negativity of all the variables yields

$$\lambda_2 > 0 \Rightarrow x_2 = 0,$$

as $\lambda_2 x_2 = 0$. Thus, we conclude that $\lambda_2 = 2$. From the equation $2x_1 + \lambda_1 = 2\lambda_3 x_1$ we conclude that if $x_1 = 0$, then $\lambda_1 = 0$. If $x_1 = \lambda_1 = 0$, then $x_1 = x_2 = 0$, and the goal function is also 0. However, if $x_1 > 0$, then $\lambda_1 = 0$ and $\lambda_3 = 1$ (follows from $2x_1 + \lambda_1 = 2\lambda_3 x_1$). Hence, by $\lambda_3\left(1 - x_1^2 - x_2^2\right) = 0$ we know that $x_2 = 0$ and $x_1 = 1$. In this situation, the goal function is $f(1, 0) = -1$ and this is the minimum of the function given the previous constraints.

Remark 8.3. For more exercises, one could use the Book “Must Have Tools for Graduate Study in Economics” by William Neilson, pages 48-51, and pages 67-70.

Part 4

Probability and Statistics

CHAPTER 9

Basic concepts in probability and statistics

9.1. Probability spaces and axioms

Every probability (down to its most basic element) starts with some kind of an experiment whose result we cannot accurately predict. The experiment has several possible outcomes, each may occur due to several parameters which we bind together as *Probability*.

To better understand the notion of probability, we start with one of the simplest experiment which is - a symmetric coin toss. Assume we have a coin with $'0'$ written on one side, and $'1'$ written on the other side. We are told that half the times the coin lands on $'0'$. Therefore, every toss of this symmetric coin is an experiment, that could end with $'0'$ with probability (w.p.) 0.5, or $'1'$ w.p. 0.5.

The set of all possible results is denoted by Ω, which is a *sample space*. A probability function $\Pr : 2^{\Omega} \to \mathbb{R}$ is a function that assigns a number $\Pr(A) \in [0,1]$ to every subset $A \subseteq \Omega$. A subset A of the sample space is call an *event*. In the example above, $\Omega = \{0,1\}$ and

$$\Pr(\{1\}) = \frac{1}{2},\ \Pr(\{2\}) = \frac{1}{2},\ \Pr(\{1,2\}) = 1,\ \Pr(\phi) = 0.$$

Another simple example is a simple symmetric dice with six faces. In this case, $\Pr(A) = \frac{|A|}{6}$ for every subset $A \subseteq \Omega$ as $\Omega = \{1,2,3,4,5,6\}$. The couple $(\Omega, \Pr)$ is called a *probability space*.

9.1.1. The Probability axioms.

There is a good reason for using the number of elements is a subset of the sample space to define the probability of that set. It is for the symmetry between the different possible results in the sample space and the basic assumptions on the probability function. There are three axioms, know as *The Probability Axioms*, or *The Kolmogorov axioms*, that we assume on $\Pr(\cdot)$:

(1) For every event $A \subseteq \Omega$, it follows that $\Pr(A) \geq 0$.
(2) $\Pr(\Omega) = 1$.
(3) Any countable sequence of disjoint[1] events $A_1, A_2, \ldots$ satisfies $\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$.

These three axioms are all the assumptions we make on the function $\Pr(\cdot)$.

Exercise 9.1. Prove the following statements:

(1) For every event A, it holds that $\Pr(A^c) = 1 - \Pr(A)$.
(2) $\Pr(\phi) = 0$.
(3) if $A \subset B$, then $\Pr(A) \leq \Pr(B)$.

Solution.

(1) Note that $A \cup A^c = \Omega$ and since A and A^c are disjoint, it follows that

$$\begin{aligned} \Pr(A) + \Pr(A^c) &= \Pr(A \cup A^c) \\ &= \Pr(\Omega) \\ &= 1 \\ \Rightarrow \Pr(A^c) &= 1 - \Pr(A). \end{aligned}$$

(2) This follows directly from the previous conclusion and the axiom $\Pr(\Omega) = 1$ when $\Omega^c = \phi$.

[1]Events $A_1, A_2, \ldots$ are disjoint if $A_i \cap A_j = \phi$ for every $i \neq j$.

(3) Define the event $C = B \setminus A$, and note that C and A are disjoint such that $C \cup A = B$. Hence,

$$\begin{aligned}\Pr(B) &= \Pr(C \cup A) \\ &= \Pr(C) + \Pr(A) \\ &\geq \Pr(A),\end{aligned}$$

since $\Pr(C) \in [0,1]$.

9.1.2. The inclusion–exclusion principle.

The inclusion-exclusion principle gives us an easy way to compute the probability of unions of events. It states that:

- For any two events A, B, it holds that

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

- For any three events A, B, C, it holds that

$$\begin{aligned}\Pr(A \cup B \cup C) &= \Pr(A) + \Pr(B) + \Pr(C) \\ &- \Pr(A \cap B) - \Pr(A \cap C) - \Pr(B \cap C) \\ &+ \Pr(A \cap B \cap C).\end{aligned}$$

EXERCISE 9.2. We toss a symmetric, six-faces dice twice.

(1) Define the probability space.
(2) Write down the event A where both tosses are identical and the event B where the sum of results equals 4.
(3) Compute $\Pr(A)$, $\Pr(B)$, and $\Pr(A \cup B)$.

Solution.

(1) The sample space is $\Omega = \{(i,j) : \ i,j = 1,2,\ldots,6\}$ and $\Pr(\omega) = 1/36$ for every $\omega \in \Omega$.
(2) $A = \{(i,i) : \ i = 1,2,\ldots,6\}$ and $B = \{(i,j) : \ i+j = 4, \ i,j = 1,2,3\}$.
(3) Using the probability axioms, we get

$$\begin{aligned}\Pr(A) &= 6 \cdot \frac{1}{36} = \frac{1}{6}, \\ \Pr(B) &= 3 \cdot \frac{1}{36} = \frac{1}{12}.\end{aligned}$$

Now we use the inclusion-exclusion principle. First,$\Pr(A \cap B) = \Pr(2,2) = \frac{1}{36}$. Thus,

$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= \frac{6}{36} + \frac{3}{36} - \frac{1}{36} = \frac{2}{9}.\end{aligned}$$

EXERCISE 9.3. There are n students in a class room. What is the probability that there exists at least one couple of students with the same birthday?

Solution. Let A be the event where there is at least one couple of students with the same birthday. We are going to compute $\Pr(A^c)$. In order for no such couple to exists, all the students must have different birthdays. Thus we need to choose n dates from 365 possible days of the year, and then distribute them to the students. The size of the sample space is $|\Omega| = 365^n$ as every student has, in general, 365 options. Therefore,

$$\begin{aligned}\Pr(A^c) &= \frac{\binom{365}{n} n!}{365^n} = \frac{365!}{365^n (365-n)!}, \\ \Rightarrow \Pr(A) &= 1 - \frac{365!}{365^n (365-n)!},\end{aligned}$$

which is more than half when $n = 23$, and about 0.994 when $n = 60$.

EXERCISE 9.4. How many solutions are there for the equation $x_1 + x_2 + \cdots + x_m = n$?

Solution. We solve this question by creating an equivalent experiment. Assume you have $m-1$ sticks and n identical balls which you place in a raw. The sticks act as partitions and the balls within two sticks is the number of balls in that partition. Note that randomly organizing these objects in a raw creates m cells (including the two outer cell created by the extreme sticks) and the number of balls states the value of that cell. This is exactly like distributing n times the number 1 into m different variables. Thus the answer to the question is $\binom{n+m-1}{n}$, or $\binom{n+m-1}{m-1}$.

EXERCISE 9.5. In a box there are 100 bulbs among which 6 are defected. We choose 5 randomly without putting them back in the box.

(1) What is the probability that we will have exactly 2 working bulbs?
(2) What is the probability that we will have at least 2 working bulbs?

Solution.

(1) Let $A_i$be the event where we have exactly i working bulbs. The probability of A_2 is

$$\Pr(A_2) = \frac{\binom{94}{2}\binom{6}{3}}{\binom{100}{5}}.$$

(2) We can sum up disjoint events and get

$$\begin{aligned} \Pr\left(\bigcup_{i=2}^{5} A_i\right) &= \sum_{i=2}^{5} \Pr(A_i) \\ &= \frac{1}{\binom{100}{5}} \sum_{i=2}^{5} \binom{94}{i}\binom{6}{5-i}. \end{aligned}$$

9.2. Conditional probability

The idea behind conditional probability is to give changes in probability when the information we have changes. For example, take a basic experiment of tossing twice a fair six-faces dice. We know that every combination (i,j) when $i,j \in 1,\ldots,6$ have the same $1/36$ probability to be realized. Now assume that someone told us that the sum of the two tosses is at least 10, now what is the probability of a couple (i,j) to be realized? Well, in this case, we can intuitively say that

$$\Pr((i,j)\,|i+j \geq 10) = \begin{cases} \frac{1}{6}, & \text{if } (i,j) = (4,6), \\ \frac{1}{6}, & \text{if } (i,j) = (6,4), \\ \frac{1}{6}, & \text{if } (i,j) = (5,5), \\ \frac{1}{6}, & \text{if } (i,j) = (5,6), \\ \frac{1}{6}, & \text{if } (i,j) = (6,5), \\ \frac{1}{6}, & \text{if } (i,j) = (6,6), \\ 0, & \text{otherwise.} \end{cases}$$

Note that we used symmetry to compute the non-zero probabilities.

DEFINITION 9.1. **(Conditional probability)** The probability of event A conditional on event B such that $\Pr(B) > 0$ is

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)}.$$

PROPOSITION 9.1. *For any event B with positive probability, the conditional probability $\Pr(\cdot|B)$ is a probability. That is, it sustains the three probability axioms.*

PROOF. We need to prove that $\Pr(\cdot|B)$ sustains the three probability axioms. First,

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \geq 0,$$

as both the numerator and the denominator are non-negative. Second,

$$\Pr(\Omega|B) = \frac{\Pr(\Omega \cap B)}{\Pr(B)} = \frac{\Pr(B)}{\Pr(B)} = 1.$$

And last, let $A_1, A_2, \ldots$ be a sequence of disjoint events. Then

$$\begin{aligned}
\Pr\left(\bigcup_i A_i|B\right) &= \frac{\Pr\left(\left(\bigcup_i A_i\right) \cap B\right)}{\Pr(B)} \\
&= \frac{\Pr\left(\bigcup_i (A_i \cap B)\right)}{\Pr(B)} \\
&= \frac{\sum_i \Pr(A_i \cap B)}{\Pr(B)} \\
&= \sum_i \frac{\Pr(A_i \cap B)}{\Pr(B)} \\
&= \sum_i \Pr(A_i|B),
\end{aligned}$$

as required. □

EXERCISE 9.6. An urn contains 10 white balls, 5 yellow balls, and 10 black balls. You take out a ball at random and it turns out that it is not black.

(1) What is the probability that it is yellow?
(2) What is the probability that it is white?

Solution.

(1) $\Pr(Y|B^c) = \frac{\Pr(Y \cap B^c)}{\Pr(B^c)} = \frac{\Pr(Y)}{\Pr(B^c)} = \frac{\frac{5}{25}}{\frac{15}{25}} = \frac{1}{3}$.
(2) $\Pr(W|B^c) = 1 - \Pr(Y|B^c) = \frac{2}{3}$.

EXAMPLE 9.1. Assume that a patient can either be healthy, an event denoted by H, or he can have a certain condition, an event denoted by U. He takes a test to determine his status. The probabilities are given in the following table:

	Positive test	Negative test	Overall
H	0.001	0.987	0.988
U	0.010	0.002	0.012
Overall	0.011	0.989	

we can see that condition U is rare, only 0.012 percent of the population have it (that is, 12 people out of every 1000 people on average).

(a) Assuming that the test was positive, what is the probability of actually having the condition?
(b) Assuming that the test was negative, what is the probability of actually having the condition?

These question are based on the idea of conditional probabilities. First, denote the event that the test was positive, meaning the person has the condition, by P, and the event where the test is negative is denoted N. We need to compute the following probabilities:

$$\Pr(U|P) = ?\ ,\ \Pr(U|N) = ?$$

Using conditional probability,

$$\begin{aligned}\Pr(U|P) &= \frac{\Pr(U\cap P)}{\Pr(P)} = \\ &= \frac{0.010}{0.011} = \frac{10}{11} \approx 0.91, \\ \Pr(U|N) &= \frac{\Pr(U\cap N)}{\Pr(N)} = \\ &= \frac{0.002}{0.989} = \frac{2}{989} \approx 0.002.\end{aligned}$$

This means that a false positive (getting a positive test when you do not have the condition) occurs w.p. 0.09. Almost 10% of the times! On the other hand, a false negative occur only 0.2% of the times.

9.2.1. Bayes' Law.

Bayes' law is basically a simple law of conditional probability that helps us to transform the relation between the event in question and the event we condition on. This law is easily derived from the definition of conditional probability: For any two events A, B both with positive probabilities,

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)}.$$

Usually in economics, we call $\Pr(A)$ the *prior* (probability), as this is the probability we have before we get additional information. The probability $\Pr(A|B)$ is called the *posterior*, as it is known to us only after event B is given.
Although Bayes' law is simple and very important in economics (game theory, finance and so on), people do not always follow it. Consider the example given by Kahneman and Tversky in 1973 (published in *Psychological Review*).

EXAMPLE 9.2. **(Kahneman and Tversky, 1973)** Some subjects are told that a group consists of 70 lawyers and 30 engineers. The rest of the subjects are told that the group has 30 lawyers and 70 engineers. All subjects were then given the following description:
Dick is a 30 year old man. He is married with no children. A man of high ability and high motivation, he promises to be quite successful in his field. He is well liked by his colleagues.
Subjects were then asked to judge the probability that Dick is an engineer. Subjects in both groups said that it is about 0.5, ignoring the prior information. Note that the new information is uninformative and irrelevant, so

$$\Pr(\text{engineer}|\text{ new information}) = \Pr(\text{engineer}),$$

or in other words, $\Pr(B|A) = \Pr(B)$. According to Bayes rule the posterior should be the same as the prior, $\Pr(A|B) = \Pr(A)$.

9.2.2. Law of total probability.

The law of total probability is a very important aspect in probability theory. It helps us to compute the probability of may events in the same manner a probability tree helps with the computation. In fact, the law of total probability is a mathematical, or more accurately an algebraic, way to write down a probability tree.

DEFINITION 9.2. **(The law of total probability)** Assume that $B_1, B_2, \ldots$ is a countable sequence (finite or infinite) of disjoint events that forms a partition of the sample space Ω. That is, $B_i \cap B_j = \phi$ and $\bigcup_i B_i = \Omega$. Then, for every event A it follows

$$\Pr(A) = \sum_i \Pr(A|B_i)\Pr(B_i).$$

This law states the one can break down every event to a countable set of smaller events, compute the probability of each conditional event and take its product with the probability of the event, on which we conditioned. This is basically the same thing we do with probability trees. First we write down all the edges (that are the different options B_i). Then we compute the probability that the event A occurs in every edge. And lastly we sum-up the probabilities of all the edges.

EXERCISE 9.7. We toss two fair, six-faces dices until the sum of the two is higher than 10.

(1) What is the probability that the number of tosses will be less than 10?
(2) What is the probability that we will get a sum of 12 before we get a sum of 11?

Solution.

(1) Let A be the required event. In order for the number of tosses to be less than 10, we need no more than 9 repetitions. Let us compute the probability of A^c. The event A^c states that there are at least 10 repetitions, which means that we need to fail 9 straight times. The probability of getting a sum that is higher than 10 is $\frac{3}{36} = \frac{1}{12}$. Thus, the probability of failing 9 straight times is $\left(\frac{11}{12}\right)^9$. Hence, $\Pr(A) = 1 - \left(\frac{11}{12}\right)^9$.
(2) We solve this question by conditioning on the first experiment. Let X be the outcome of the first experiment and let A be the event where we get a sum of 12 before we get a sum of 11.

$$\begin{aligned}\Pr(A) &= \Pr(A|X=11)\Pr(X=11)\\ &+ \Pr(A|X=12)\Pr(X=12)\\ &+ \Pr(A|X\neq 11,12)\Pr(X\neq 11,12)\\ &= 0\cdot\frac{2}{36}+1\cdot\frac{1}{36}+\Pr(A)\cdot\frac{33}{36},\end{aligned}$$

where we use symmetry between the tosses to conclude that if the sum of the first toss in neither 11 nor 12, then the probability of A remains the same. Thus,

$$\begin{aligned}\frac{3}{36}\Pr(A) &= \frac{1}{36},\\ \Pr(A) &= \frac{1}{3}.\end{aligned}$$

EXERCISE 9.8. In a casino we have two slot machines. One of them has a 0.4 probability of winning and the other has a 0.2 probability of winning. A person chooses the following strategy: he picks a machine at random, if he wins, he plays another game, otherwise he plays the next game in the other machine.

(1) What is the probability of losing both games?
(2) What is the probability of winning exactly one game?

Solution.

(1) We need to condition on the machine that the person choose. Let A be the event where he choose the machine with the higher probability and denote the event where he chose the other machine by B. We use W and L to denote a win or a lose, respectively.

$$\begin{aligned}\Pr(L,L) &= \Pr(L,L|A)\Pr(A)+\Pr(L,L|B)\Pr(B)\\ &= \frac{3}{5}\cdot\frac{4}{5}\cdot\frac{1}{2}+\frac{4}{5}\cdot\frac{3}{5}\cdot\frac{1}{2}\\ &= \frac{12}{25}.\end{aligned}$$

(2) Let us compute the probability of winning both games.

$$\begin{aligned}\Pr(W,W) &= \Pr(W,W|A)\Pr(A)+\Pr(W,W|B)\Pr(B)\\ &= \left(\frac{2}{5}\right)^2\cdot\frac{1}{2}+\left(\frac{1}{5}\right)^2\cdot\frac{1}{2}\\ &= 0.1.\end{aligned}$$

Thus, the probability of winning exactly one game is $1-0.5-0.1=0.4$.

EXERCISE 9.9. A binary signal passes through a noisy system.

- If the signal is $'0'$, there is a probability of e_0 that is goes through as $'1'$.
- If the signal is $'1'$, there is a probability of e_1 that is goes through as $'0'$.

answer the following questions:

(1) Assuming the we sent out $'0'$ w.p. p, and $'1'$ w.p. $1-p$. What is the probability that the signal received is accurate?
(2) What is the probability that the signal $'1011'$ reaches its destination correctly?
(3) Assuming that we submit every signal three times and it is interpreted according to the majority rule. What is the probability that a signal of $'0'$ will be decoded accurately?
(4) Given the probabilities in the previous questions 1 and 3. If we received $'101'$, what are the chances that the original transmission was $'000'$?

Solution.

(1) Denote by A the event where the signal passed correctly. We use $'s_0','s_1'$ to denote the signal that were sent out and $'r_0','r_1'$ to denote the signals received.

$$\begin{aligned}\Pr(A) &= \Pr(r_0|s_0)\Pr(s_0)+\Pr(r_1|s_1)\Pr(s_1)\\ &= (1-e_0)\,p+(1-e_1)(1-p)\,.\end{aligned}$$

(2) The probability is $(1-e_1)^3(1-e_0)$.
(3) The signal will be decoded correctly if among the three received signals there are at least two $'0'$. Therefore, the probability is $(1-e_0)^3+\binom{3}{2}(1-e_0)^2e_0$.
(4) First, we wish to write down the question mathematically.

$$\Pr(s_0s_0s_0|r_1r_0r_1)=?$$

We can compute this using the law of total probability and Bayes' law.

$$\begin{aligned}\Pr(s_0s_0s_0|r_1r_0r_1) &= \frac{\Pr(r_1r_0r_1|s_0s_0s_0)\Pr(s_0s_0s_0)}{\Pr(r_1r_0r_1)}\\ &= \frac{e_0^2(1-e_0)\,p}{\Pr(r_1r_0r_1|s_0s_0s_0)\Pr(s_0s_0s_0)+\Pr(r_1r_0r_1|s_1s_1s_1)\Pr(s_1s_1s_1)}\\ &= \frac{e_0^2(1-e_0)\,p}{e_0^2(1-e_0)\,p+e_1(1-e_1)^2(1-p)}.\end{aligned}$$

Exercise 9.10. There are three chests with two drawers each. All look the same from the outside. In one chest there are two gold coins, one in each drawer, in another chest there are two silver coins and the last has one silver coin and one gold coin. We pick a chest and a drawer randomly. Given that we found a gold coin, what are the chances that the other coin is also gold?

Solution. We need to condition on the chest we chose. Denote the following events:

- G_i, choosing the chest with i gold coins when $i=0,1,2$.
- A, finding a gold coin in a random drawer we choose.

$$\begin{aligned}\Pr(G_2|A) &= \frac{\Pr(A|G_2)\Pr(G_2)}{\Pr(A)}\\ &= \frac{1\cdot(2/3)}{1/2}=\frac{2}{3}.\end{aligned}$$

Exercise 9.11. In the first group there are 17 girls and 3 boys. In the second group there are 5 girls and 10 boys. We choose two students randomly from the second group and move them to the first group. Then, we choose randomly one student from the first group. What is the probability that the last student is a boy? is a girl?

Solution. We need to condition on the students that were moved from the second group to the first. Let G_i be the event where i boys moved from the second group to the first $(i = 0, 1, 2)$.

$$
\begin{aligned}
\Pr(\text{Boy}) &= \sum_{i=0}^{2} \Pr(\text{Boy}|G_i)\Pr(G_i) \\
&= \frac{3}{22}\Pr(G_0) + \frac{4}{22}\Pr(G_1) + \frac{5}{22}\Pr(G_2) \\
&= \frac{3}{22}\cdot\frac{\binom{5}{2}}{\binom{15}{2}} + \frac{4}{22}\cdot\frac{\binom{10}{1}\binom{5}{1}}{\binom{15}{2}} + \frac{5}{22}\cdot\frac{\binom{10}{2}}{\binom{15}{2}} \\
&= \frac{3}{22}\cdot\frac{4\cdot 5}{14\cdot 15} + \frac{4}{22}\cdot\frac{5\cdot 10\cdot 2}{14\cdot 15} + \frac{5}{22}\cdot\frac{9\cdot 10}{14\cdot 15} \\
&= \frac{1}{77} + \frac{20}{77\cdot 3} + \frac{15}{77\cdot 2} = \frac{13}{66},
\end{aligned}
$$

and $\Pr(\text{Girl}) = 1 - \Pr(\text{Boy}) = \frac{53}{66}$.

Exercise 9.12. There are 3 coins, two fair coins (with equal probabilities to fall on both sides) and one coin that lands on $'H'$ w.p. 0.25. We choose one coin at random and flip it twice.

(1) What is the probability of getting exactly one $'H'$?
(2) Given we got exactly one time $'H'$, what is the probability that we chose the non-symmetric coin?

Solution.

(1) We condition on the chosen coin. Let F denote the event of a fair coin chosen, and let U denote the event of choosing the unfair coin. Denote by H_i the event of getting i times $'H'$. Therefore,

$$
\begin{aligned}
\Pr(H_1) &= \Pr(H_1|U)\Pr(U) + \Pr(H_1|F)\Pr(F) \\
&= 2\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{3} + 2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2}{3} = \frac{11}{24}.
\end{aligned}
$$

(2) Bayes' law states that

$$
\begin{aligned}
\Pr(U|H_1) &= \frac{\Pr(H_1|U)\Pr(U)}{\Pr(H_1)} \\
&= \frac{2\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{3}}{\frac{11}{24}} = \frac{3}{11}.
\end{aligned}
$$

9.2.3. The Monty Hall Problem.

This problem is based on a game show called "Lets Make a Deal" whose host for several years was Monty Hall.

Exercise 9.13. **(The Monty Hall Problem)** There are three doors, labeled A, B, and C, when behind only one of them there is a prize and nothing behind the other two (with equal probabilities for each door). The host, Monty Hall, offers a contestant the choice among three doors. Then, the host, who knows where the prize is, reveals one of the doors with no prize. The contestant needs to choose whether to stay with her original pick or switch to the other door. What should she do?

Solution. The answer is that she should take the other door. To see why, suppose she chooses door A, and that Monty reveals door B. What is the probability that the prize is behind door C given that door B was revealed? Bayes' rule says we use the formula

$$
\Pr(\text{prize in } C|\text{ open } B) = \frac{\Pr(\text{open } B|\text{ prize in } C)\Pr(\text{prize in } C)}{\Pr(\text{open } B)}.
$$

By symmetry, we know that $\Pr(\text{prize in } C) = \Pr(\text{prize in } A) = \Pr(\text{prize in } B) = 1/3$. When the prize is in C, the probability of opening door B is 1. This is due to the fact that Monty cannot open door A (as this is the door the contestant choose), and he cannot open door C, as the prize is there. Using the law of total probability we get

$$\begin{aligned}\Pr(\text{open } B) &= \Pr(\text{open } B|\text{ prize in } A)\Pr(\text{prize in } A)\\ &+ \Pr(\text{open } B|\text{ prize in } B)\Pr(\text{prize in } B)\\ &+ \Pr(\text{open } B|\text{ prize in } C)\Pr(\text{prize in } C)\\ &= \frac{1}{2}\cdot\frac{1}{3}+0\cdot\frac{1}{3}+1\cdot\frac{1}{3}=\frac{1}{2}.\end{aligned}$$

Thus

$$\begin{aligned}\Pr(\text{prize in } C|\text{ open } B) &= \frac{\Pr(\text{open } B|\text{ prize in } C)\Pr(\text{prize in } C)}{\Pr(\text{open } B)}\\ &= \frac{1\cdot\frac{1}{3}}{\frac{1}{2}}=\frac{2}{3}.\end{aligned}$$

This means that the probability of winning by switching is $2/3$, and this is true also in the case that Monty opens door C. Since A is arbitrarily, we conclude that switching guarantees a probability of $2/3$ of winning.

9.3. Independent events

We say that two events are independent if the realization of one event does not affect the probability that the other is realized.

Definition 9.3. Let A, B be two events given a probability space $(\Omega, \Pr)$. The events A and B are independent if $\Pr(A\cap B)=\Pr(A)\Pr(B)$.

Exercise 9.14. In the library there are 10 probability books, 5 with solutions. When a student comes to collect a book, he gets one at random. One book is lost. A student borrows a book and brings it back after 3 days. A week later, a different student comes and also borrows a book at random. Define the following events:

- A- the first student got a book with solutions.
- B- the second student got a book with solutions.

Are A and B independent?

Solution. Let S be the event the the lost book has solution. Thus,

$$\begin{aligned}\Pr(A) &= \Pr(A|S)\Pr(S)+\Pr(A|S^c)\Pr(S^c)\\ &= \frac{4}{9}\cdot\frac{1}{2}+\frac{5}{9}\cdot\frac{1}{2}=\frac{1}{2}.\end{aligned}$$

One can verify that the same computation hols for $\Pr(B)$. However,

$$\begin{aligned}\Pr(A\cap B) &= \Pr(A\cap B|S)\Pr(S)+\Pr(A\cap B|S^c)\Pr(S^c)\\ &= \frac{4^2}{81}\cdot\frac{1}{2}+\frac{5^2}{81}\cdot\frac{1}{2}=\frac{41}{162}>\frac{1}{2}\cdot\frac{1}{2}=\Pr(A)\Pr(B),\end{aligned}$$

which means that the event are dependent. The intuition behind this result is interesting. Although the students do not take book at the same time, the fact the one either got a book with solution or without, affects the probability that the other will get a book with solutions. To make this more intuitive, try think of what would happen had we repeated the process for many times. Would we then know what kind of book was lost?

Exercise 9.15. Two fair dices are tossed. Consider the events:

- A, the result in the first dice is odd.
- B, the result in the second dice is odd.
- C, the sum of the results is odd.

(1) Prove that every two events from the above are independent.

(2) Are A, B, C independent? That is, do they satisfy the equality $\Pr(A \cap B \cap C) = \Pr(A)\Pr(B)\Pr(C)$?

Solution.

(1) This is proven by straightforward computation.

$$\Pr(A) = \Pr(B) = \frac{1}{2},$$

and $\Pr(C) = \frac{1}{2}$ since C occurs once one dice is odd and the other is even. The chances for that are 0.25 and there are two options (for each of the coins), so the probability is half.

$$\begin{aligned}
\Pr(A \cap B) &= \frac{1}{4} = \Pr(A)\Pr(A), \\
\Pr(A \cap C) &= \Pr(C|A)\Pr(A) \\
&= \frac{1}{2} \cdot \frac{1}{2} = \Pr(C)\Pr(A),
\end{aligned}$$

and the same result is reached when A is replaced by B. To conclude, every two events are independent.

(2) The three events are dependent. Why?

$$\Pr(A \cap B \cap C) = 0,$$

since the sum of two odd numbers is even. this means that $\Pr(A \cap B \cap C) \neq \Pr(A)\Pr(B)\Pr(C)$.

EXERCISE 9.16. **(Polya's Urn)** Assume there is an urn with 8 balls, 5 black balls and 3 white ones. Whenever we take out a ball randomly, we put it back in along with 4 more of the same color.

(1) What is the probability that the first ball we take out is black?
(2) What is the probability that the second ball we take out is black?
(3) What is the probability that the 100^{th} ball we take out is black?

Solution.

(1) The probability is $\frac{5}{8}$.
(2) Denote the event where the i-th ball is black by B_i.

$$\begin{aligned}
\Pr(B_2) &= \Pr(B_2|B_1)\Pr(B_1) + \Pr(B_2|B_1^c)\Pr(B_1^c) \\
&= \frac{9}{12} \cdot \frac{5}{8} + \frac{5}{12} \cdot \frac{3}{8} = \frac{5}{8},
\end{aligned}$$

and we got the same result.

(3) Since we do not want to consider all the possible cases up to the 100th stage, we try using symmetry. Assume that the balls are numbered such that the black balls have numbers from 1 to 5 and the white ones have numbers from 6 to 8. Now assume that whenever a ball is taken out, we put it back in along with four other balls with the same number and color. By symmetry, we know that the probability of taking out a ball with a digit $i = 1, \ldots, 8$ is $\frac{1}{8}$, since all the numbers have the same chance of being picked. This means that the chances of taking out a ball with a number 1 till 5, i.e., a black ball, is $\frac{5}{8}$.

EXERCISE 9.17. A disease hits 1 in every $20,000$ people. A diagnostic test is 95% accurate, that is, the test is positive for 95% of people with the disease, and negative for 95% of the people who do not have the disease. Max just tested positive for the disease. What is the probability he has it?

Solution. We use the previously-defined notation where H means healthy, U means unhealthy, P means a positive test, and N means a negative test.

$$\begin{aligned}
\Pr(U|P) &= \frac{\Pr(P|U)\Pr(U)}{\Pr(P)} \\
&= \frac{\frac{95}{100}\cdot\frac{1}{20{,}000}}{\Pr(P|U)\Pr(U)+\Pr(P|H)\Pr(H)} \\
&= \frac{\frac{95}{100}\cdot\frac{1}{20{,}000}}{\frac{95}{100}\cdot\frac{1}{20{,}000}+\frac{5}{100}\cdot\frac{19{,}999}{20{,}000}} \\
&= \frac{95}{95+5\cdot 19,999} = 0.000949.
\end{aligned}$$

In words, although the test came in positive, the chances of actually being sick is less than 0.01%.

Exercise 9.18. You have data that sorts individuals into occupations and age groups. There are three occupations: doctor, lawyer, and entrepreneur. There are two age categories: below 40 (young) and above 40 (old). You wanted to know the probability that an old person is an entrepreneur. Your grad student misunderstands you, though, and presents you with the following information:

- 20% of the sample are doctors and 30% are entrepreneurs;
- 40% of the doctors are young;
- 20% of the entrepreneurs are young;
- 70% of the lawyers are young.

Find the probability that the an old person is an entrepreneur.

Solution. Define the following events:

- A person is an entrepreneur - E.
- A person is a doctor - D.
- A person is a lawyer - L.
- A person is young - Y.
- A person is old - O.

We need to compute $\Pr(E|O)$.

$$\begin{aligned}
\Pr(E|O) &= \frac{\Pr(O|E)\Pr(E)}{\Pr(O)} \\
&= \frac{0.8\cdot 0.3}{0.8\cdot 0.3+0.6\cdot 0.2+0.3\cdot 0.5} \\
&= \frac{24}{24+12+15} = \frac{8}{17} \approx 0.47.
\end{aligned}$$

CHAPTER 10

Probability functions and probability density functions

10.1. Discrete Random variables

A random variable (RV) $X : \Omega \to \mathbb{R}$ is a function such that for every result $\omega \in \Omega$ of the experiment, get a number $X(\omega)$. For example, we toss two fair dices and X is the sum of the results. Clearly, X could be every natural number from 2 to 12 and the probability that X equals such a number changes. E.g., $\Pr(X = 2) = \frac{1}{36}$ as it occurs only when both dices land on 1, while $\Pr(X = 7) = \frac{1}{6}$ (verify this!). Usually we use random variables as tolls to compute many things, such as averages etc. A random variable that can get a countable set of values is called a *discrete random variable*. The set of values that any RV can get is referred to as its *support*.

10.1.1. Distributions.

Consider an experiment where we toss a fair dice. Let X be the result of the experiment. That is, X gets any natural number from 1 to 6 with equal probabilities. The distribution P_X of X is a function that assigns every value $k \in \mathbb{R}$, the probability that $X = k$. Specifically, $P_X : \mathbb{R} \to [0, 1]$ is a function form the real numbers to $[0, 1]$ such that $P_X(k) = \Pr(x = k)$. In this example,

$$P_X(k) = \begin{cases} \frac{1}{6}, & \text{if } k = 1, 2, 3, 4, 5, 6, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 10.1. **(Distribution)** For every discrete RV X, the distribution P_X is a function from $\mathbb{R}$ to $[0, 1]$, such that $P_X(k) = \Pr(X = k)$ for every $k \in \mathbb{R}$.

Exercise 10.1. We toss two fair dices and X is the sum of their results. Find the support and the distribution of X.

Solution. The support of X is the set $\{n \in \mathbb{N} : \ 2 \leq n \leq 12\}$. The distribution of X is

$$\begin{aligned} \Pr(X = 2) &= \frac{1}{36} \quad , \quad \Pr(X = 3) = \frac{2}{36}, \\ \Pr(X = 4) &= \frac{3}{36} \quad , \quad \Pr(X = 5) = \frac{4}{36}, \\ \Pr(X = 6) &= \frac{5}{36} \quad , \quad \Pr(X = 7) = \frac{6}{36}, \\ \Pr(X = 8) &= \frac{5}{36} \quad , \quad \Pr(X = 9) = \frac{4}{36}, \\ \Pr(X = 10) &= \frac{3}{36} \quad , \quad \Pr(X = 11) = \frac{2}{36}, \\ \Pr(X = 12) &= \frac{1}{36}. \end{aligned}$$

10.1.2. Cumulative distribution function.

An important function regarding any RV is the *cumulative distribution function*, CDF, denoted by F_X for every RV X.[1] The CDF F_X is defined from $\mathbb{R}$ to $\mathbb{R}$ such that for every $k \in \mathbb{R}$, $F_X(k) = \Pr(X \leq k)$. That is, it sums up the values of the distribution of X up until the value k, hence its name "cumulative distribution function".

The CDF has three properties:

(1) $\lim_{k\to\infty} F_x(k) = 1$ and $\lim_{k\to-\infty} F_x(k) = 0$.

[1]When the random variable is clear from the context, the CDF is denoted by F.

(2) $F_x(\cdot)$ is monotone non-decreasing.
(3) $F_x(\cdot)$ is right-continuous.

EXERCISE 10.2. We toss three fair dices and X is the maximal value of their results.

(1) Find the CDF of X.
(2) Use the previous result to find the distribution of X.

Solution. Note that this is a symmetric sample space, and $|\Omega| = 6^3 = 216$.

(1) The CDF of X is

$$\begin{aligned} F_X(k) &= \Pr(X \le k) \\ &= \frac{1}{216} \cdot \begin{cases} 0, & k < 1, \\ 1^3, & 1 \le k < 2, \\ 2^3, & 2 \le k < 3, \\ 3^3, & 3 \le k < 4, \\ 4^3, & 4 \le k < 5, \\ 5^3, & 5 \le k < 6, \\ 6^3, & k > 6. \end{cases} \end{aligned}$$

(2) We can use the previous results to compute the distribution of X.

$$\begin{aligned} P_X(k) &= \Pr(X = k) \\ &= \Pr(X \le k) - \Pr(X \le k-1) \\ &= F_X(k) - F_X(k-1) \\ &= \begin{cases} \frac{k^3-(k-1)^3}{216}, & k = 1,2,3,4,5,6, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

10.1.3. Common discrete distributions.

There are several commonly-used RVs, and therefore their distributions are well known and categorized. As these random variables are common, we tend to remember their properties. Most of the times we relate to the distributions rather than thee RVs themselves. Nevertheless, we will define the RVs first, and their distributions are deduced directly.

10.1.3.1. *Bernoulli distribution.* First we define the most basic random variable, which is a *Bernoulli random variable* X with parameter $p \in [0,1]$, denoted by $X \sim B(p)$. The RV X is basically an indicator that equals 1 if an experiment succeeds (occurs w.p. p), or 0, otherwise. A *Bernoulli distribution* with parameter p is

$$P_X(k) = \begin{cases} p, & k = 1, \\ 1-p, & k = 0, \end{cases}$$

10.1.3.2. *Binomial distribution.* The next random variable is the *Binomial random variable* with parameters (n,p), denoted by $X \sim Bin(n,p)$. The RV X counts the number of success in n independent experiments where each experiment succeeds w.p. p. Therefore, A *Binomial distribution* with parameters n,p is

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k},$$

for every $k = 0, 1, \ldots, n$.

10.1.3.3. *Geometric distribution.* Another very common random variable is the *Geometric* random variable with parameter $p \in [0,1]$, denoted by $X \sim G(p)$. The RV X count the number of experiments in a sequence of independent Bernoulli experiments (all with parameter $p \in [0,1]$) needed to reach the first success. Its *Geometric distribution* is

$$P_X(k) = (1-p)^{k-1} p,$$

for every $k \in \mathbb{N}$.

10.1.3.4. *Poisson distribution.* One random variable that can be observed in many places in our everyday lives is *the Poisson random variable.* This random variable counts the number of events in a given time frame (or, given some other units), when there is a fixed average rate of occurrences. A Poisson random variable X with parameter λ, denoted by $X \sim Pois(\lambda)$, can equal any non-negative integer, under the following *Poisson distribution*,

$$P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!},$$

when $k = 0, 1, 2, \ldots$. One can prove that the Poisson distribution is derived from the binomial distribution when $n \to \infty$ and $np = \lambda$. Note that λ is the average rate of occurrences.

10.1.3.5. *Uniform distribution.* The last discrete common RV we are going to discuss is the uniformly distributed RV. The *uniform random variable* X with parameters $a, b \in \mathbb{Z}$, denoted $X \sim U[a, b]$ equals every number between a and b with equal probability. That is, *the uniform distribution* is

$$P_X(k) = \frac{1}{b-a+1} \; \forall k = a, a+1 \ldots, b.$$

There are several more common distribution, such as *the negative binomial*, and *the hyper-geometric* and more. Nevertheless we will focus on these ones, and the others could be found in any probability textbook.

EXERCISE 10.3. A drunk person is moving one step to the right w.p. p and one step to the left w.p. $1-p$. If he moves to the right, then w.p. q he slides two steps to the left. Let X be his location after n steps. Find the distribution of X.

Solution. Let Y be the number of successful steps to the right, without sliding back to the left. $Y \sim Bin(n, p(1-q))$ and

$$n = Y + \text{steps right with slide} + \text{steps left}.$$

Since a step right with a slide of two to the left is basically a step left, then

$$X = Y - \text{steps right with slide} - \text{steps left}.$$

Combining this two equations yields

$$X = 2Y - n.$$

Thus,

$$\begin{aligned}
\Pr(X = k) &= \Pr(2Y - n = k) \\
&= \Pr\left(Y = \frac{k+n}{2}\right) \\
&= \binom{n}{\frac{k+n}{2}} [p(1-q)]^{\frac{n+k}{2}} [1 - p(1-q)]^{\frac{n-k}{2}},
\end{aligned}$$

when $k \in \mathbb{Z},\ k \in [-n, n]$, and $\frac{k+n}{2} \in \mathbb{N}$.

EXERCISE 10.4. We toss two coins until one shows "Heads" and the other "Tails". One coin shows "Heads" w.p. p, while the other lands on "Heads" w.p. q. Let X be the number of rounds.

(1) Find the distribution of X.
(2) What are the chances that the first coin (i.e., the one w.p. p) will show "Heads" in the last round?

Solution.

(1) We have a sequence of i.i.d. Bernoulli experiments, where a success is reached w.p. $p(1-q)+q(1-p)$ in each. Thus, $X \sim G(p(1-q)+q(1-p))$.
(2) The fact that it is the last round implies that one coin fell on "Heads" and the other on "Tails". Denote this event by A. Denote the event where the first coin shows "Heads" in the last round by B.

$$\begin{aligned}\Pr(B|A) &= \frac{\Pr(B\cap A)}{\Pr(A)} \\ &= \frac{p(1-q)}{p(1-q)+q(1-p)}.\end{aligned}$$

Exercise 10.5. A drunk person moves either one step right or one step left with equal probabilities and independently of past steps. After a hundred steps, he is located at $+10$ to the right of his starting point. Find the distribution of his first step.

Solution. The first step could be either right, $+1$, or left, -1. Let p be the probability that the first step is $+1$. We know that until the $100th$ round, he made 55 steps right and 45 steps left. As all the steps are symmetric, when randomly choosing a first step from this collection there is a 0.55 probability of getting $+1$, and a probability of 0.45 of getting -1.

10.2. Continuous Random variables

A continuous random variable is a random variable whose support (that is, the set of values he can take) is uncountable and is piece-wise convex. In other words, the support is given by a union of non-degenerate intervals. For example, take the interval $[0,1]$ and choose uniformly a point within this interval. Denote its value by X. In this case, X is a continuous RV distributed uniformly on $[0,1]$.

10.2.1. Cumulative probability distribution and density functions.

Although continuous RVs are random variables, the fact that they can take an uncountable number of values makes them very different from discrete random variables. First, the probability that a single specific point is chosen is 0. In fact, when discussing continuous RVs, we do not discuss the distribution as we previously studied, but we use a different function to describe the RVs, which is the density function.

A continuous random variable X has a CDF, just as a discrete RV, and its definition is the same, $F_X(t)=\Pr(x\le t)$. However, when it comes to continuous RVs, the CDF is not only continuous, but also continuously differentiable, C^1. For this reason, we can discuss its derivative $f_x(t)=\frac{dF_x(t)}{dt}$ which is an integrable, non-negative function whose integral on $\mathbb{R}$ equals 1. This function is called *the probability density function* (PDF) of X. The PDF has the following properties:

(1) $f_X(t)\ge 0$ for every $t\in\mathbb{R}$.
(2) ${}_{-\infty}\int^{\infty} f_X(t)\,dt=1$.
(3) ${}_{-\infty}\int^{k} f_X(t)\,dt=F_X(k)$ for every $k\in\mathbb{R}$.

The support of a continuous RV is the smallest closed set such that $\{t\in\mathbb{R}:\ f(t)>0\}$.

10.2.2. Common continuous distributions.

There are a few continuous RVs whose classes are common and are important to remember.

10.2.2.1. *Uniform distribution.* Let $a<b$ be two real numbers. The uniform (and continuous) random variable between a and b, denoted by $X\sim U(a,b)$ is a RV that can take any value in $[a,b]$ and whose density function is constant. That is,

$$f_X(t)=\frac{1}{b-a},\ \forall t\in[a,b],$$

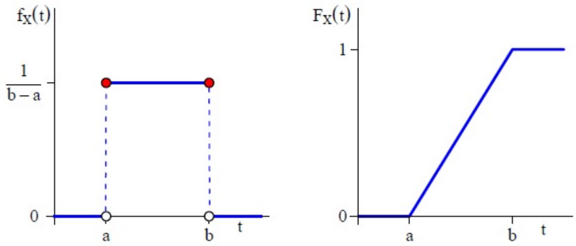

FIGURE 10.2.1. PDF and CDF of a uniform distribution

and equals 0, otherwise. Hence, the CDF of X is

$$\begin{aligned} F_X(t) &= \Pr(X \leq t) \\ &= \int_{-\infty}^{t} f_X(s)\, ds \\ &= \begin{cases} 0, & \text{if } t < a, \\ \frac{t-a}{b-a}, & \text{if } a \leq t \leq b, \\ 1, & \text{if } t > b. \end{cases} \end{aligned}$$

The graphs of these functions are presented in Figure 10.2.1.

10.2.2.2. *Exponential distribution.* An exponential RV count the time between events that occur according to a Poisson distribution. That is, if a Poisson RV count the number of events in a given time, the exponential RV counts the time between events. In some sense, these RV are equivalent as each one of them defines the other. The parameter of this RV (and distribution) is λ, and the density function is given by

$$f_X(t) = \lambda e^{-\lambda t}, \ \forall t \geq 0,$$

and 0, otherwise. Its CDF is

$$\begin{aligned} F_X(t) &= \Pr(X \leq t) \\ &= \int_{-\infty}^{t} f_X(s)\, ds \\ &= \begin{cases} 0, & \text{if } t < 0, \\ 1 - e^{-\lambda t}, & \text{if } t \geq 0. \end{cases} \end{aligned}$$

It is denoted $X \sim Exp(\lambda)$.

10.2.2.3. *Normal distribution.* The *normal distribution* is probably the most common and used distribution of all. A RV with a normal distribution has two parameter, μ and σ^2, which will be discussed later on. It is denoted by $X \sim N(\mu, \sigma^2)$ and its density function is

$$f_X(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(t-\mu)^2}{2\sigma^2}}, \ \forall t \in \mathbb{R}.$$

When $\mu = 0$ and $\sigma = 1$, the normal distribution is called a *standard distribution*. The CDF of the standard distribution is denoted by $\Phi(\cdot)$. The graphs of these functions are presented in Figure 10.2.2.

REMARK 10.1. We do not have an explicit function for the CDF of the normal distribution. The reason for that is that such a representation does not exists and the function is computed numerically. Therefore, in order to find the relevant probabilities we use the normal distribution table. One important property of the CDF of the normal distribution is symmetry. That is,

$$\Phi(-t) = 1 - \Phi(t).$$

This property is quite useful when using the table of the normal distribution, since it contains only positive values and do not related to cases where $t < 0$.

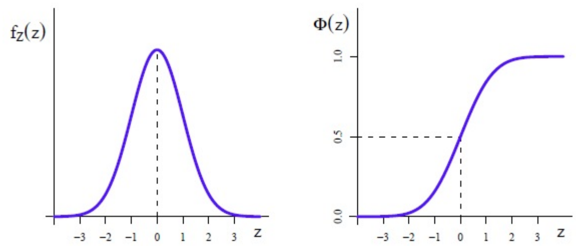

FIGURE 10.2.2. PDF and CDF of a normal distribution

10.2.3. Examples and Exercises.

EXAMPLE 10.1. **(Second-Price auctions)**[2] A simple application of Leibniz's rule comes from auction theory. A first-price sealed bid auction has bidders submit bids simultaneously to an auctioneer who awards the item to the highest bidder who then pays his bid. This is a very common auction form. A second-price sealed bid auction has bidders submit bids simultaneously to an auctioneer who awards the item to the highest bidder, just like before, but this time the winning bidder pays the second-highest price. To model the second-price auction, suppose that there are n bidders and that bidder i values the item being auctioned at v_i, which is independent of how much everyone else values the item. Bidders do not know their opponents valuations, but they do know the probability distribution of the opponents valuations. Bidder i must choose his bid b_i.

Let $F_i(b)$ be the probability that the highest other bid faced by i, that is, the highest bid except for b_i, is no larger than b. Then $F_i(b)$ is a probability distribution function, and its density function is $f_i(b)$. Bidder i's expected payoff is

$$V_i(b_i) = \int_0^{b_i} (v_i - b) f_i(b) db.$$

Lets interpret this function. Bidder i wins if his bid is the highest bid, which occurs if the highest other bid is between 0 (the lowest possible bid) and his own bid b_i. If the highest other bid is above b_i bidder i loses and gets a payoff of zero. This is why the integral is taken over the interval $[0, b_i]$. If bidder i wins he pays the highest other bid b, which is distributed according to the density function $f_i(b)$. His surplus if he wins is $v_i - b$, his value minus how much he pays.

Bidder i chooses the bid b_i to maximize his expected payoff $V_i(b_i)$. Since this is a maximization problem we should find the first-order condition:

$$V_i'(b_i) = \frac{d}{db_i} \int_0^{b_i} (v_i - b) f_i(b) db = 0.$$

Notice that we are differentiating with respect to b_i, which shows up only as the upper endpoint of the integral. Using Leibniz's rule we can evaluate this first-order condition:

$$\begin{aligned} 0 &= \frac{d}{db_i} \int_0^{b_i} (v_i - b) f_i(b) db \\ &= \int_0^{bi} \frac{\partial}{\partial b_i} [(v_i - b) f_i(b)] db + \frac{db_i}{db_i} \cdot (v_i - b_i) f_i(b_i) - \frac{d0}{db_i} \cdot (v_i - 0) f_i(0). \end{aligned}$$

The first term is zero because $(v_i - b) f_i(b)$ is not a function of b_i, and so the partial derivative is zero. The second term reduces to $(v_i - b_i) f_i(b_i)$ because $\frac{db_i}{db_i}$ is simply one. The third term is zero because the derivative $\frac{d0}{db_i} = 0$. This leaves us with the first-order condition

$$0 = (v_i - b_i) f_i(b_i)$$

Since density functions take on only non-negative values, the first-order condition holds when $v_i - b_i = 0$, or $b_i = v_i$. In a second-price auction the bidder should bid his value.

[2]Taken from *MUST-HAVE MATH TOOLS FOR GRADUATE STUDY IN ECONOMICS* by William Neilson, Department of Economics University of Tennessee – Knoxville September 2009 © web.utk.edu/~wneilson/mathbook.pdf

This result makes sense intuitively. Let b_i be bidder is bid, and let b denote the highest other bid. Suppose first that bidder i bids more than his value, so that $b_i > v_i$. If the highest other bid is in between these, so that $v_i < b < b_i$, bidder i wins the auction but pays $b > v_i$ more than his valuation. He could have avoided this by bidding his valuation, v_i. If $b > b_i$ or $b < v_i$, then bidding b_i does not matter compared to bidding v_i.

Now suppose that bidder i bids less than his value, so that $b_i < v_i$. If the highest other bid is between these two, so that $b_i < b < v_i$, bidder i loses the auction and gets nothing. But if he had bid his value he would have won the auction and paid $b < v_i$, and so he would have been better off. Again, if $b < b_i$ or $b > v_i$, then bidding b_i does not matter compared to bidding v_i. Thus, the best thing for him to do is bid his value.

Definition 10.2. **(First-order stochastic dominance)** For every two lotteries F and G such that $G(x) \geq F(x)$ in every x, we say that F stochastically dominates G (also called, *first-order stochastic dominance*).

Exercise 10.6. Let $U(a, b)$ denote the uniform distribution over the interval $[a, b]$. Find conditions on a and b that guarantee that $U(a, b)$ (first-order) stochastically dominates $U(0, 1)$.

Solution. First we need to write down the lottery G, which is the CDF of $U(0, 1)$ and the lottery F, which is the CDF of $U(a, b)$.

$$G(x) = \begin{cases} 0, & x < 0, \\ \frac{x-0}{1-0}, & 0 \leq x \leq 1, \\ 1 & x > 1, \end{cases}$$

and

$$F(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1 & x > b. \end{cases}$$

Now, we need to observe $G(x) - F(x)$. Clearly, if $a < 0$, then taking every $a < x < \min(0, b)$ yields

$$G(x) - F(x) = 0 - F(x) < 0.$$

Thus, $a \geq 0$. If $b < 1$, then every $\max(b, 0) < x < 1$ yields

$$G(x) - F(x) = G(x) - 1 < 0.$$

This implies that $a \geq 0$ and $b \geq 1$. Let us write $G(x) - F(x)$explicitly.

$$\begin{aligned} G(x) - F(x) &= \begin{cases} 0, & x < 0, \\ x, & 0 \leq x \leq a, \\ x - \frac{x-a}{b-a}, & a \leq x \leq 1, \\ 1 - \frac{x-a}{b-a}, & 1 \leq x \leq b, \\ 0 & x > b, \end{cases} \\ &= \begin{cases} 0, & x < 0, \\ x, & 0 \leq x \leq a, \\ \frac{x(b-1)+a(1-x)}{b-a}, & a \leq x \leq 1, \\ \frac{b-x}{b-a}, & 1 \leq x \leq b, \\ 0 & x > b, \end{cases} \end{aligned}$$

and the result holds.

EXERCISE 10.7. The time X it takes to serve a client is distributed exponentially with $\lambda = 0.5$. In order to improve the service time, the employees are being trained to speak faster. The new service time is $Y = \frac{1}{2}X + \frac{1}{10}$. Find the PDF of Y.

Solution. To find the density of Y, we first need to find the CDF of Y.

$$
\begin{aligned}
F_Y(t) &= \Pr(Y \le t) \\
&= \Pr\left(\frac{1}{2}X + \frac{1}{10} \le t\right) \\
&= \Pr\left(X \le 2t - \frac{1}{5}\right) \\
&= F_X\left(2t - \frac{1}{5}\right) \\
&= \begin{cases} 0, & \text{if } 2t - \frac{1}{5} < 0, \\ 1 - e^{-\lambda\left(2t - \frac{1}{5}\right)}, & \text{if } 2t - \frac{1}{5} \ge 0, \end{cases} \\
&= \begin{cases} 0, & \text{if } t < \frac{1}{10}, \\ 1 - e^{-t + \frac{1}{10}}, & \text{if } t \ge \frac{1}{10}. \end{cases}
\end{aligned}
$$

And the density is

$$
f_Y(t) = \begin{cases} 0, & \text{if } t < \frac{1}{10}, \\ e^{-t + \frac{1}{10}}, & \text{if } t \ge \frac{1}{10}. \end{cases}
$$

EXERCISE 10.8. The time X an economist exits his work place is distributed $U(7, 9)$. The driving time home is distributed $Y = 1 + \frac{1}{X}$. Find the density of Y.

Solution. We need to find the CDF of Y and then the density. Note that $Y \in \left(1 + \frac{1}{9}, 1 + \frac{1}{7}\right)$,

$$
\begin{aligned}
F_Y(t) &= \Pr(Y \le t) \\
&= \Pr\left(1 + \frac{1}{X} \le t\right) \\
&= \Pr\left(\frac{1}{t-1} \le X\right) \\
&= 1 - \Pr\left(X < \frac{1}{t-1}\right) \\
&= 1 - \begin{cases} 0, & \text{if } \frac{1}{t-1} < 7, \\ \frac{\frac{1}{t-1} - 7}{9-7}, & \text{if } 7 \le \frac{1}{t-1} \le 9, \\ 1, & \text{if } 9 < \frac{1}{t-1}, \end{cases} \\
&= \begin{cases} 1, & \text{if } 1 + \frac{1}{7} < t, \\ \frac{9t-10}{2(t-1)}, & \text{if } 1 + \frac{1}{9} \le t \le 1 + \frac{1}{7}, \\ 0, & \text{if } t < 1 + \frac{1}{9}. \end{cases}
\end{aligned}
$$

The density is

$$
f_Y(t) = \begin{cases} \frac{1}{2(t-1)^2}, & \text{if } t \in \left[\frac{10}{9}, \frac{8}{7}\right], \\ 0, & \text{otherwise.} \end{cases}
$$

CHAPTER 11

Joint distributions - probability functions in several variables

11.1. Joint distributions

Until now we discussed one-dimensional RVs. Clearly, this is not the general case. One can think of RVs in several dimensions as a vector where each coordinate is a one-dimensional RV. For example, consider an experiment where two fair dices are tossed. Let X be the sum of the results and let Y be the maximal result between the two. The support of (X,Y) is $\left\{(i,j)\in\mathbb{N}^2:\ i=2,3\ldots,12,\ j=1,2,\ldots,6\right\}$. The *joint distribution* $P_{X,Y}$ of (X,Y) is a function such that for every $k,l\in\mathbb{R}^2$,

$$P_{X,Y}\left(k,l\right)=\Pr\left(X=k,Y=l\right).$$

The *marginal distribution* of X is the distribution of X, and it could be derived from the joint distribution by summing over the support of Y,

$$\Pr\left(X=k\right)=\sum_{l}\Pr\left(X=k,Y=l\right).$$

The same holds for the marginal distribution of Y. This is true for the discrete case. For the continuous case, we can discuss the joint distribution of (X,Y) when relating the the CDF

$$F_{X,Y}\left(k,l\right)=\Pr\left(X\leq k,Y\leq l\right).$$

In this case, the density function is $f_{X,Y}\left(k,l\right)=\frac{\partial F_{X,Y}(k,l)}{\partial k\partial l}$. Similarly to the discrete case, we can use integration to compute the marginal density of X by

$$f_X\left(k\right)=_{-\infty}\int^{\infty}f_{X,Y}\left(k,l\right)dl,$$

and

$$F_X\left(x\right)=_{-\infty}\int_{-\infty}^{x}\int^{\infty}f_{X,Y}\left(k,l\right)dldk.$$

The same holds for Y.
The *conditional distribution* of X given $Y=l$ is given by

$$f_{X|Y}\left(k|l\right)=\frac{f_{X,Y}\left(k,l\right)}{f_Y\left(l\right)},$$

when the RVs are continuous, and

$$P_{X|Y}\left(k|l\right)=\frac{P_{X,Y}\left(k,l\right)}{P_Y\left(l\right)},$$

when the RVs are discrete.
In case A is an event with positive probability, then *conditional distribution* of a continuous random variable X given A is

$$f_{X|A}\left(k\right)=\frac{f_X\left(k\right)}{\Pr\left(A\right)}\cdot\mathbf{1}_{\{\{X=k\}\subseteq A\}}.$$

11.2. Independent random variables

DEFINITION 11.1. Two random variables X,Y are independent if for every $k,l\in\mathbb{R}$,they satisfy the equality

$$f_{X,Y}\left(k,l\right)=f_X\left(k\right)f_Y\left(l\right),$$

in case the RVs are continuous; And the equality

$$P_{X,Y}(k,l) = P_X(k)\,P_Y(l),$$

in case the RVs are discrete.

Note that we the above equalities must hold for every k and every l, and not just for some values. The intuition behind the definition of independence of RVs is as follows. When two RVs are independent, the fact that one of them is fixed, does not change the distribution of the other. Basically, when we fix one of them, the other maintains the same values and probabilities it already had, and is not affected in any way.

EXERCISE 11.1. The joint distribution of X, Y is given in the following table:

$X\backslash Y$	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$
$X=5$	0.01	0.03	0.17	0.00	0.00
$X=20$	0.03	0.05	0.04	0.2	0.12
$X=30$	0.11	0.04	0.02	0.07	0.11

Answer the following questions:

(1) Find $\Pr(X=30)$, and $\Pr(X\in[5,20], Y\in[2,4])$.
(2) Given that $X\geq 20$, find the probability that $Y\leq 2$.
(3) Determine whether the events $A=\{X\leq 20\}$ and $B=\{Y\in\{1,4\}\}$ are independent.

Solution.

(1) The probability $\Pr(X=30)$ could be computed by summing over all the values of Y when $X=30$.

$$\begin{aligned}\Pr(X=30) &= \sum_{y=1}^{5}\Pr(X=30, Y=y)\\ &= 0.11+0.04+0.02+0.07+0.11\\ &= 0.35.\end{aligned}$$

$$\begin{aligned}\Pr(X\in[5,20], Y\in[2,4]) &= 0.03+0.17+0.05+0.04+0.2\\ &= 0.49.\end{aligned}$$

(2) We need to compute $\Pr(Y\leq 2|X\geq 20)$.

$$\begin{aligned}\Pr(Y\leq 2|X\geq 20) &= \frac{\Pr(Y\leq 2, X\geq 20)}{\Pr(X\geq 20)}\\ &= \frac{0.05+0.03+0.04+0.11}{\Pr(X=20)+\Pr(X=30)}\\ &= \frac{0.23}{0.79}=\frac{23}{79}.\end{aligned}$$

(3) We need to find the probability of the following events: A, B, and $A\cap B$.

$$\begin{aligned}\Pr(A) &= \Pr(X\leq 20)\\ &= 1-\Pr(X=30)\\ &= 0.65,\end{aligned}$$

$$\begin{aligned}\Pr(B) &= \Pr(Y=1)+\Pr(Y=4)\\ &= 0.15+0.27=0.42,\end{aligned}$$

$$\begin{aligned}\Pr(A\cap B) &= \Pr(Y\in\{1,4\}, X\leq 20)\\ &= 0.04+0.2=0.24.\end{aligned}$$

Hence,

$$\Pr(A \cap B) = 0.24 \neq 0.273 = 0.42 \cdot 0.65 = \Pr(A) \cdot \Pr(B),$$

which means they are dependent.

EXERCISE 11.2. The joint distribution of X, Y is given in the following table:

$X \backslash Y$	$Y = 1$	$Y = 2$	$Y = 3$	$Y = 4$
$X = 1$	0.02	0.02	0.21	0.02
$X = 2$	0.03	0.01	0.05	0.06
$X = 3$	0.01	0.01	0.01	0.06
$X = 4$	0.00	0.05	0.00	0.12
$X = 5$	0.12	0.06	0.00	0.14

Answer the following questions:

(1) Which event is more likely $A = \{X \in [3, 4]\}$ or $B = \{Y \leq 2, X = 5\}$?
(2) Find the probability that $Y \neq 3$.
(3) Compute $\Pr(Y = 2 | X = 5)$ and $\Pr(X \geq 3 | Y \in \{1, 4\})$.
(4) Determine whether the events $A = \{X \in \{1, 3\}\}$ and $B = \{Y \in \{1, 2, 4\}\}$ are independent?

Solution.

(1) We find the probability of both events by summing over the relevant cells.

$$\begin{aligned} \Pr(A) &= 0.01 + 0.01 + 0.01 + 0.06 + 0.05 + 0.12 = 0.26, \\ \Pr(B) &= 0.12 + 0.06 = 0.18, \end{aligned}$$

and so event A is more likely to occur.

(2) Using the complement of $Y \neq 3$ yields

$$\begin{aligned} \Pr(Y \neq 3) &= 1 - \Pr(Y = 3) \\ &= 1 - (0.21 + 0.05 + 0.01) \\ &= 0.73. \end{aligned}$$

(3) These probabilities are easily computed using the definition of conditional probability.

$$\begin{aligned} \Pr(Y = 2 | X = 5) &= \frac{\Pr(Y = 2 \cap X = 5)}{\Pr(X = 5)} \\ &= \frac{0.06}{0.06 + 0.12 + 0.14} \\ &= \frac{0.06}{0.32} = \frac{3}{16}. \end{aligned}$$

$$\begin{aligned} \Pr(X \geq 3 | Y \in \{1, 4\}) &= \frac{\Pr(X \geq 3 \cap Y \in \{1, 4\})}{\Pr(Y \in \{1, 4\})} \\ &= \frac{0.45}{0.58} = \frac{45}{58}. \end{aligned}$$

(4) We need to find the probability of the following events: A, B, and $A \cap B$.

$$\Pr(A) = \Pr(X \in \{1, 3\}) = 0.36,$$

$$\Pr(B) = \Pr(Y \in \{1, 2, 4\}) = 0.73,$$

$$\Pr(A \cap B) = \Pr(Y \in \{1, 2, 4\}, X \in \{1, 3\}) = 0.14.$$

Hence,

$$\Pr(A \cap B) = 0.14 \neq 0.2628 = 0.73 \cdot 0.36 = \Pr(A) \cdot \Pr(B),$$

which means the events are not independent.

CHAPTER 12

Moments

Moments are types of averages of RVs. As every random variable X is a stochastic function (gets values with certain probabilities), one can discuss its average value, or the average value of X^2 and so on. A moment of degree k is the average value of X^k and we will now see how it is computed.

12.1. Expectation

The expected value of a random variable X, denoted $\mathbf{E}[X]$ is

$$\mathbf{E}[X] = \begin{cases} \sum_k k \Pr(X = k), & X \text{ is discrete,} \\ \int t f_X(t)\, dt, & X \text{ is continuous.} \end{cases}$$

In general, the expectation of a RV is the weighted average of the values he gets, times the probabilities of getting these values.
The expectation has the important property of linearity. For every RVs x, y and a real number $c \in \mathbb{R}$,

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y] \ , \ \mathbf{E}[cX] = c\mathbf{E}[X].$$

Exercise 12.1. Find the expected value of the following random variables:

(1) $X \sim U[a,b]$.
(2) $X \sim Bin(n,p)$.
(3) $X \sim Pois(\lambda)$.
(4) $X \sim U(a,b)$.
(5) $X \sim Exp(\lambda)$.
(6) $X \sim N(0,1)$.

Solution. We will compute these expected values one by one.[1]

$$\begin{aligned} X &\sim U[a,b] \\ \mathbf{E}[X] &= \sum_{k=a}^{b} k \Pr(X = k) \\ &= \sum_{k=a}^{b} \frac{k}{b-a+1} \\ &= \frac{1}{b-a+1} \sum_{k=a}^{b} k \\ &= \frac{1}{b-a+1} \cdot \frac{(b+a)(b-a+1)}{2} \\ &= \frac{b+a}{2}. \end{aligned}$$

[1]You can find different and additional computations of expected values online or in most probability textbooks.

$$\begin{aligned}
X &\sim Bin\left(n,p\right) \\
\mathbf{E}\left[X\right] &= \sum_{k=0}^{n} k\Pr\left(X=k\right) \\
&= \sum_{k=1}^{n} k\Pr\left(X=k\right) \\
&= \sum_{k=1}^{n} k\binom{n}{k}p^{k}\left(1-p\right)^{n-k} \\
&= np\sum_{k=1}^{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}p^{k-1}\left(1-p\right)^{n-k} \\
&= np\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}\left(1-p\right)^{n-k} \\
&= np\sum_{k=0}^{n-1}\binom{n-1}{k}p^{k}\left(1-p\right)^{n-k-1} \\
&= np\left(p+1-p\right)^{n-1} = np,
\end{aligned}$$

and the last line is due to the binomial formula.

$$\begin{aligned}
X &\sim Pois\left(\lambda\right) \\
\mathbf{E}\left[X\right] &= \sum_{k=1}^{\infty} k\Pr\left(X=k\right) \\
&= \sum_{k=1}^{\infty} ke^{-\lambda}\frac{\lambda^{k}}{k!} \\
&= \lambda e^{-\lambda}\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!} \\
&= \lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k}}{k!} \\
&= \lambda e^{-\lambda}e^{\lambda} = \lambda,
\end{aligned}$$

when the last line follows from the Taylor expansion of e^{λ}.

$$\begin{aligned}
X &\sim U\left(a,b\right) \\
\mathbf{E}\left[X\right] &= \int_{\mathbb{R}} kf_{X}\left(k\right)dk \\
&= \int_{a}^{b} k\frac{1}{b-a}dk \\
&= \frac{1}{b-a}\cdot\frac{b^{2}-a^{2}}{2} \\
&= \frac{b+a}{2}.
\end{aligned}$$

$$\begin{aligned} X &\sim Exp(\lambda) \\ \mathbf{E}[X] &= \int_{\mathbb{R}} k f_X(k)\,dk \\ &= \int_0^{\infty} k\lambda e^{-\lambda k} dk \\ &= \lambda\left[\left(k\cdot\frac{e^{-\lambda k}}{-\lambda}\right)|_0^{\infty} + \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda k} dk\right] \\ &= \int_0^{\infty} e^{-\lambda k} dk \\ &= -\frac{1}{\lambda}e^{-\lambda k}|_0^{\infty} = \frac{1}{\lambda}. \end{aligned}$$

$$\begin{aligned} X &\sim N(0,1) \\ \mathbf{E}[X] &= \int_{\mathbb{R}} kdk \\ &= \int_{-\infty}^{\infty} k\frac{1}{\sqrt{2\pi}}e^{\frac{-k^2}{2}} dk \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} ke^{\frac{-k^2}{2}} dk \\ &= 0, \end{aligned}$$

as $e^{\frac{-k^2}{2}}$ is an even function (symmetric around $k = 0$) and k is an odd function (asymmetric around $k = 0$), hence $ke^{\frac{-k^2}{2}}$ is an odd function and its integral on a symmetric interval around $k = 0$ is 0.

EXAMPLE 12.1. **(Choice between lotteries)**[2] Suppose that an individual is choosing between two lotteries. Lotteries are just probability distributions, and the individual wishes to maximize its expected utility with an increasing utility function $u(x)$, where x is an mount of money given in the lottery. Assume that the amount of money in every lottery is bounded between a and b such that $b > a$.

EXERCISE 12.2. Given a lottery F (in other words, F is a CDF over amount of money), what is the objective function of the decision maker? That is, what is the function that the decision maker wishes to maximize?

Solution. The objective function of the decision maker is his expected utility given by

$$\int_a^b u(x)F'(x)dx,$$

where a is the lowest possible payoff from the lottery, b is the highest possible payoff, $u(x)$ is a utility function defined over amounts of money, and $F'(x)$ is the density function corresponding to the CDF $F(x)$. We assume that the decision maker prefers to get more money than less, meaning u is non-decreasing and $u'(x) \geq 0$.

EXERCISE 12.3. The individual can choose between lottery $F(x)$ and lottery $G(x)$. Assume that $F(x) \leq G(x)$ for every x. Prove that the decision maker would choose F over G.

PROOF. In order to answer this question, we prove that

$$\int_a^b u(x)F'(x)dx - \int_a^b u(x)G'(x)dx \geq 0,$$

[2]Taken from *MUST-HAVE MATH TOOLS FOR GRADUATE STUDY IN ECONOMICS* by William Neilson, Department of Economics University of Tennessee – Knoxville September 2009 © web.utk.edu/~wneilson/mathbook.pdf

which implies that the by choosing F instead of G, the decision maker is maximizing its objective function. Thus, using integration by parts

$$\begin{aligned}
\int_a^b u(x)F'(x)dx - \int_a^b u(x)G'(x)dx &= \int_a^b u(x)\left[F'(x) - G'(x)\right]dx \\
&= \left(u(x)\left[F(x) - G(x)\right]\right)|_a^b - \int_a^b u'(x)\left[F(x) - G(x)\right]dx \\
&= u(b)\left[F(b) - G(b)\right] - u(a)\left[F(a) - G(a)\right] - \int_a^b u'(x)\left[F(x) - G(x)\right]dx \\
&= u(b)\left[1 - 1\right] - u(a)\left[0 - 0\right] - \int_a^b u'(x)\left[F(x) - G(x)\right]dx \\
&= \int_a^b u'(x)\left[G(x) - F(x)\right]dx \\
&\geq 0,
\end{aligned}$$

when the last inequality follows from the non-negativity of $u'(x)$ and $G(x) - F(x)$ in every x. □

EXERCISE 12.4. Your annul income depends on the number of products X you are able to sell. There are 5 companies interested in your product, each will eventually buy it with probability 0.25 (independently of the other firms). The retail price of the product is 1 million dollars and your annul costs are fixed, and equal \$500,000. Find the expected annual income.

Solution. Let Y denote the annual income. Clearly, $Y = 10^6 \cdot X - 5 \cdot 10^5$. In addition, $X \sim Bin(5, 0.25)$, which implies $\mathbf{E}[X] = \frac{5}{4}$. Hence, by the linearity of the expectation we get

$$\begin{aligned}
\mathbf{E}[Y] &= 10^6 \cdot \mathbf{E}[X] - 5 \cdot 10^5 \\
&= 10^6 \cdot \frac{5}{4} - 5 \cdot 10^5 \\
&= 5 \cdot 10^5 \left(\frac{10}{4} - 1\right) \\
&= 750,000.
\end{aligned}$$

12.1.1. The expected value of a function of a RV.

In many cases, we do not know the distribution of the RV whose expected value we wish to compute. Although this could be problematic, there are cases in which the computation is not too difficult. Fix a random variable X and let $Y = g(X)$ be a function of X. That is, $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function, and so $Y = g(X)$ is a new random variable. In this case, the expected value of Y is

$$\mathbf{E}[Y] = \begin{cases} \sum_k g(k) \Pr(X = k), & X \text{ is discrete}, \\ \int g(t) f_X(t)\, dt, & X \text{ is continuous}. \end{cases}$$

Note that the distributions, P_X and f_X, are taken w.r.t. the random variable X, and not Y. Thus, in cases that the distribution of X is known and Y is a known function of X, finding the expected value $\mathbf{E}[Y]$ is just a direct computation.

12.1.2. Conditional expectation.

Another value we should consider is the *conditional expectation*. The conditional expectation of X given $Y = l$ is

$$\mathbf{E}[X|Y = l] = \begin{cases} \sum_k k P_{X|Y}(k|l), & \text{when the RVs are discrete}, \\ \int_{\mathbb{R}} k f_{X|Y}(k|l)\, dk, & \text{when the RVs are continuous}. \end{cases}$$

12.1.3. The law of iterated expectation.

Conditional expectation is useful just as the law of total probability. Its helps us computing the expected value of a RV by conditioning on another RV.

THEOREM 12.1. **(The law of iterated expectation)** *Let X, Y be two RVs with finite expected values such that the expected value of $X|Y$ exists and is also finite. Then,*

$$\mathbf{E}[X] = \mathbf{E}[\mathbf{E}[X|Y]].$$

Note that $\mathbf{E}[X|Y]$ is a function of Y. Thus, the expected value of $\mathbf{E}[X|Y]$ is taken over the values of Y.

EXERCISE 12.5. There are two random variables X, Y with a joint probability distribution given in the following table. Show that the law of iterated expectation works for the random variable X.

$\Pr(X = k, Y = l)$	$l = 1$	$l = 2$	$\Pr(X = k)$
$k = 1$	0.1	0.3	0.4
$k = 2$	0.2	0.1	0.3
$k = 3$	0.1	0.2	0.3
$\Pr(Y = l)$	0.4	0.6	

Solution. Lets begin by computing the expected value of X and the expected value of $X|Y = l$ for every $l = 1, 2$.

$$\begin{aligned}
\mathbf{E}[X] &= 1 \cdot 0.4 + 2 \cdot 0.3 + 3 \cdot 0.3 = 1.9. \\
\mathbf{E}[X|Y = 1] &= 1 \cdot \frac{0.1}{0.4} + 2 \cdot \frac{0.2}{0.4} + 3 \cdot \frac{0.1}{0.4} = 2, \\
\mathbf{E}[X|Y = 2] &= 1 \cdot \frac{0.3}{0.6} + 2 \cdot \frac{0.1}{0.6} + 3 \cdot \frac{0.2}{0.6} = \frac{11}{6}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{E}[\mathbf{E}[X|Y]] &= \mathbf{E}[X|Y = 1]\Pr(Y = 1) + \mathbf{E}[X|Y = 2]\Pr(Y = 2) \\
&= 2 \cdot 0.4 + \frac{11}{6} \cdot 0.6 \\
&= \frac{8}{10} + \frac{11}{10} = 1.9 = \mathbf{E}[X],
\end{aligned}$$

as required.

EXAMPLE 12.2. **(Calculating the benefit of a search)**[3] Consider the following search process. A consumer, Max, wants to buy a particular digital camera. He goes to a store and looks at the price. At that point he has three choices: (i) buy the camera at that store, (ii) go to another store to check its price, or (iii) go back to a previous store and buy the camera there. Stores draw their prices P independently from the distribution $F(p)$ given by

$$P = \begin{cases} 170, & \text{w.p. } 0.1, \\ 180, & \text{w.p. } 0.4, \\ 190, & \text{w.p. } 0.3, \\ 200, & \text{w.p. } 0.2. \end{cases}$$

EXERCISE 12.6. We want to answer the following question: If the lowest price so far is q, what is the expected benefit from checking one more store?

Solution. Lets begin by answering this in the most straightforward way possible. Suppose that $q = 200$, so that the lowest price found so far is the worst possible price. If Max searches one more time there is a 10% chance of finding a price of \$170 and saving \$30, a 40% chance of finding a price of \$180 and saving \$20, a 30% chance of finding a price of \$190 and saving only \$10, and a 20% chance of finding another store that charges the highest possible price of \$200, in which case the savings are zero.
The expected saving is

$$0.1 \cdot 30 + 0.4 \cdot 20 + 0.3 \cdot 10 + 0.2 \cdot 0 = 14.$$

[3]Taken from *MUST-HAVE MATH TOOLS FOR GRADUATE STUDY IN ECONOMICS* by William Neilson, Department of Economics University of Tennessee – Knoxville September 2009 © web.utk.edu/~wneilson/mathbook.pdf

When $q = 200$, the expected benefit of search is \$14. \
Now suppose that $q = 190$, so that the best price found so far is \$190. Max has a 10% chance of finding a price of \$170 and saving \$20, a 40% chance of finding a price of \$180 and saving \$10, a 30% chance of finding the same price and saving nothing, and a 20% chance of finding a higher price of \$200, in which case he also saves nothing. The expected saving is

$$0.1 \cdot 20 + 0.4 \cdot 10 + 0.3 \cdot 0 + 0.2 \cdot 0 = 6.$$

When the best price found so far is $q = 190$, the expected benefit of search is \$6.
Finally, suppose that $q = 180$. Now there is only one way to improve, which comes by finding a store that charges a price of \$170, leading to a \$10 saving. The probability of finding such a store is 10%, and the expected saving from search is \$1.
So now we know the answers, and lets use these answers to figure out a general formula, specifically one involving conditional expectations. Note that when Max finds a price of p and the best price so far is q, his benefit is $q - p$, in the case the new price p is lower than the old price q. Otherwise the benefit is zero because he would be better off buying the item at a store hes already found. This "if" statement lends itself to a conditional expectation. In particular, the "if" statement pertains to the conditional expectation $\mathbf{E}[q - P|P < q]$, where the expectation is taken over the random variable P. This expression tells us what the average benefit is, provided that the benefit is non-negative. The actual expected benefit is

$$\Pr(P < q)\, \mathbf{E}[q - P|P < q],$$

which is the probability that the benefit is positive times the expected benefit conditional on the benefit being positive. Lets make sure this works using the above example. In particular, lets look at $q = 190$. The expected benefit is

$$\begin{aligned}
\Pr(P < 190)\, \mathbf{E}[190 - P|P < 190] &= 0.5 \cdot [(190 - 180) \Pr(P = 180|P < 190)] \\
&+ 0.5 \cdot [(190 - 170) \Pr(P = 170|P < 190)] \\
&= 0.5 \left[10 \cdot \frac{\Pr(180)}{\Pr(P < 190)} + 20 \cdot \frac{\Pr(170)}{\Pr(P < 190)} \right] \\
&= 10 \cdot 0.4 + 20 \cdot 0.1 = 6,
\end{aligned}$$

which is exactly what we found before.
The conditional expectation lets us work with more complicated distributions. Suppose that prices are drawn independently from the uniform distribution over the interval $[150, 200]$. Let the corresponding distribution function be $F(p)$ and the density function be $f(p)$. The expected benefit from searching at another store when the lowest price so far is q is

$$\begin{aligned}
\Pr(P < q)\, \mathbf{E}[q - P|P < q] &= F(q) \int_{150}^{q} [q - p] \frac{f(p)}{F(q)} dp \\
&= \int_{150}^{q} [q - p] f(p) dp.
\end{aligned}$$

To see why this works, look at the top line. The probability that $P < q$ is simply $F(q)$, because that is the definition of the distribution function. That gives us the first term on the right-hand side. For the second term, note that we are taking the expectation of $q - p$, so that term is in brackets. To find the conditional expectation, we multiply by the conditional density which is the density of the random variable p divided by the probability that the conditioning event $(P < q)$ occurs. We take the integral over the interval $[150, q]$ because outside of this interval the value of the benefit is zero. When we multiply the two terms on the right-hand side of the top line together, we find that the $F(q)$ term cancels out, leaving us with the very simple bottom line. Using it we can find the net benefit of searching at one more store when the best price so far is \$182.99:

$$\int_{150}^{q}[q-p]f(p)dp=\int_{150}^{182.99}[182.99-p]\cdot\frac{1}{50}dp=10.883.$$

12.2. Variance

The next moment we will discuss is the second moment. The *variance,* $\mathbf{V}(x)$ (or, $\mathbf{Var}(X)$) of a random variable X is $\mathbf{V}(x)=\mathbf{E}\left[(X-\mathbf{E}[X])^2\right]$. In words, it is the expected value of the new RV $(X-\mathbf{E}(X))^2$, which defines the distance between a RV and its average. Therefore, the variance is the average distance of a random variable and its expectation. Another way to denote the variance of a random variable X is σ_X^2 when $\sigma_X=\sqrt{\mathbf{V}(X)}$ is called the *standard deviation* of X.
The variance us a parameter that measures deviations of a RV from the expected value. When a RV gets most of its value in the same area, e.g., a constant RV, then its variance would be low. For example, assume that $X=c$, a constant RV. In this case, $\mathbf{E}[X]=c$, and

$$\mathbf{V}(X)=\mathbf{E}\left[(X-c)^2\right]=\mathbf{E}\left[(c-c)^2\right]=0.$$

On the other hand, if X has a wide range of values, then its variance will grow significantly.
The variance is not linear, however there are a few important properties it poses:

(1) For every random variable X and real numbers $a,b\in\mathbb{R}$, $\mathbf{V}(aX+b)=a^2\mathbf{V}(X)$.
(2) If X,Y are independent, then $\mathbf{V}(X+Y)=\mathbf{V}(X)+\mathbf{V}(Y)$.

In most cases, the easiest way to compute the variance $\mathbf{V}(X)$ of a random variable X is by computing its expected value $\mathbf{E}[X]$, and the expected value of X^2. These two values generate the variance in the following manner,

$$\begin{aligned}\mathbf{V}(X) &= \mathbf{E}\left[(X-\mathbf{E}[X])^2\right]\\ &= \mathbf{E}\left[X^2-2X\mathbf{E}[X]+(\mathbf{E}[X])^2\right]\\ &= \mathbf{E}\left[X^2\right]-2\mathbf{E}[X\mathbf{E}[X]]+(\mathbf{E}[X])^2\\ &= \mathbf{E}\left[X^2\right]-2(\mathbf{E}[X])^2+(\mathbf{E}[X])^2\\ &= \mathbf{E}\left[X^2\right]-(\mathbf{E}[X])^2,\end{aligned}$$

when the forth equality follows from the fact that $\mathbf{E}[X]$ is a number which implies $\mathbf{E}[X\mathbf{E}[X]]=\mathbf{E}[X]\mathbf{E}[X]$ as the expectation is linear.

12.2.1. Variance of known distributions.

In the following table you can find a list of known distributions and their variances (along with other properties we discussed). You could derive these values yourselves by straightforward computation (or found it online, or in most probability textbooks).

Distribution	Support	probability \ density	Expectation	Variance
$B(p)$	$k=0,1$	$P_X(1)=p$	p	$p(1-p)$
$Bin(n,p)$	$k=1,2,\ldots,n$	$P_X(k)=\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
$G(p)$	$k\in\mathbb{N}\setminus\{0\}$	$P_X(k)=(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
$Pois(\lambda)$	$k\in\mathbb{N}\cup\{0\}$	$P_X(k)=e^{-\lambda}\frac{\lambda^k}{k!}$	λ	λ
$U[a,b]$	$k\in a,a+1,\ldots,b$	$P_X(k)=\frac{1}{b-a+1}$	$\frac{b+a}{2}$	$\frac{(b-a+1)^2-1}{12}$
$U(a,b)$	$k\in[a,b]$	$f_X(k)=\frac{1}{b-a}$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$
$Exp(\lambda)$	$k\in[0,\infty)$	$f_X(k)=\lambda e^{-\lambda k}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$N(\mu,\sigma^2)$	$k\in\mathbb{R}$	$f_X(k)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{\frac{-(k-\mu)^2}{2\sigma^2}}$	μ	σ^2

EXERCISE 12.7. Suppose that the random variable X has the following distribution:

$$P_X(k) = \begin{cases} -14, & \text{w.p. } 0.02, \\ -6, & \text{w.p. } 0.10, \\ -2, & \text{w.p. } 0.15, \\ 2, & \text{w.p. } 0.40, \\ 4, & \text{w.p. } 0.23, \\ 7, & \text{w.p. } 0.10. \end{cases}$$

Find the mean and variance of X.

Solution. We can compute both values directly.

$$\begin{aligned} \mathbf{E}[X] &= -14 \cdot 0.02 - 6 \cdot 0.1 - 2 \cdot 0.15 \\ &+ 2 \cdot 0.4 + 4 \cdot 0.23 + 7 \cdot 0.1 = 1.24. \end{aligned}$$

$$\begin{aligned} \mathbf{V}[X] &= \mathbf{E}\left[(X - 1.24)^2\right] \\ &= (-14 - 1.24)^2 \cdot 0.02 + (-6 - 1.24)^2 \cdot 0.1 + (-2 - 1.24)^2 \cdot 0.15 \\ &+ (2 - 1.24)^2 \cdot 0.4 + (4 - 1.24)^2 \cdot 0.23 + (7 - 1.24)^2 \cdot 0.1 = 16.762. \end{aligned}$$

EXERCISE 12.8. Consider a random variable X with PDF $f_X(k) = 2k$ on $[0, 1]$.

(1) Find the CDF of X, and prove that it is a CDF.
(2) Find the mean and variance of X.

Solution.

(1) We can find the CDF by integrating the density function.

$$\begin{aligned} F_X(k) &= \int_{-\infty}^{k} f_X(t)\, dt \\ &= \begin{cases} \int_{-\infty}^{k} 0dt, & \text{if } k < 0, \\ \int_{0}^{k} 2tdt, & \text{if } k \in [0, 1], \\ \int_{0}^{1} 2tdt, & \text{if } k > 1, \end{cases} \\ &= = \begin{cases} 0, & \text{if } k < 0, \\ k^2, & \text{if } k \in [0, 1], \\ 1, & \text{if } k > 1. \end{cases} \end{aligned}$$

We can see the it follows the required properties as $\lim_{k\to\infty} F_x(k) = 1$, $\lim_{k\to-\infty} F_x(k) = 0$, and $F_x(\cdot)$ is a continuous, monotone, non-decreasing function.

(2) A direct computation yields

$$\begin{aligned} \mathbf{E}[X] &= \int_0^1 k \cdot 2kdk \\ &= 2\int_0^1 k^2 dk \\ &= \frac{2}{3}, \end{aligned}$$

$$\begin{aligned}\mathbf{E}\left[X^2\right] &= \int_0^1 k^2 \cdot 2k dk \\ &= 2\int_0^1 k^3 dk \\ &= \frac{1}{2},\end{aligned}$$

and so

$$\begin{aligned}\mathbf{V}\left[X\right] &= \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2 \\ &= \frac{1}{2} - \frac{4}{9} \\ &= \frac{1}{18}.\end{aligned}$$

Exercise 12.9. Prove that for every random variable X and real numbers $a, b \in \mathbb{R}$,

$$\mathbf{V}\left(aX+b\right) = a^2 \mathbf{V}\left(X\right).$$

Solution. Fix a random variable X and real numbers $a, b \in \mathbb{R}$. Note that $\mathbf{E}\left[ax+b\right] = a\mathbf{E}\left[X\right] + b$, thus

$$\begin{aligned}\mathbf{V}\left(aX+b\right) &= \mathbf{E}\left[\left(aX+b-\mathbf{E}\left[ax+b\right]\right)^2\right] \\ &= \mathbf{E}\left[\left(aX+b-a\mathbf{E}\left[X\right]-b\right)^2\right] \\ &= \mathbf{E}\left[a^2\left(X-\mathbf{E}\left[X\right]\right)^2\right] \\ &= a^2\mathbf{E}\left[\left(X-\mathbf{E}\left[X\right]\right)^2\right] \\ &= a^2\mathbf{V}\left(X\right).\end{aligned}$$

Exercise 12.10. Suppose that the random variable X takes the values 6 and y with equal probabilities (and only these values). Find the derivative $\frac{d\mathbf{V}(X)}{dy}$.

Solution. First we need to compute the variance $\mathbf{V}\left(X\right)$.

$$\begin{aligned}\mathbf{E}\left[X\right] &= \frac{6+y}{2}, \\ \mathbf{E}\left[X^2\right] &= \frac{36+y^2}{2}, \\ \mathbf{V}\left[X\right] &= \mathbf{E}\left[X^2\right] - \left(\mathbf{E}\left[X\right]\right)^2 \\ &= \frac{36+y^2}{2} - \frac{36+12y+y^2}{4} \\ &= \frac{72+2y^2-36-12y-y^2}{4} \\ &= \frac{36+y^2-12y}{4} \\ &= \left(\frac{y-6}{2}\right)^2.\end{aligned}$$

Thus,

$$\frac{d\mathbf{V}\left(X\right)}{dy} = 2\left(\frac{y-6}{2}\right)\cdot\frac{1}{2} = \frac{y-6}{2}.$$

12.2.2. Covariance.

The covariance of two random variables X, Y is defined by

$$\mathbf{Cov}\left(X,Y\right) = \mathbf{E}\left[XY\right] - \mathbf{E}\left[X\right]\mathbf{E}\left[Y\right].$$

The covariance is significant when trying to compute the variance of a sum of RVs. Consider the random variables $X_1, \ldots, X_n$, then

$$\mathbf{V}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathbf{V}(X_i) + \sum_{i=1}^{n}\sum_{j\neq i} \mathbf{Cov}(X_i X_j)$$
$$= \sum_{i=1}^{n} \mathbf{V}(X_i) + 2\sum_{i=1}^{n}\sum_{j<i} \mathbf{Cov}(X_i X_j).$$

The covariance has the following properties. Fix two random variables X, Y.

(1) For every real numbers $a, b \in \mathbb{R}$, it holds that $\mathbf{Cov}(aX + b, Y) = a\mathbf{Cov}(X, Y)$.
(2) $\mathbf{Cov}(X, Y) = \mathbf{Cov}(Y, X)$.
(3) $\mathbf{Cov}(X, X) = \mathbf{V}(X)$.

DEFINITION 12.1. **(Correlation)** Two random variables X, Y are *uncorrelated* if $\mathbf{Cov}(X, Y) = 0$.

In other words, two RVs are uncorrelated if $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$. Note that two uncorrelated RVs, does not mean they are independent. Nevertheless, as the following lemma states, the converse is true.

LEMMA 12.1. *If X, Y are independent, then they are also uncorrelated.*

12.2.3. The correlation coefficient.

One of the main uses of the covariance is the *"Pearson's correlation coefficient"*, also know as *"the correlation coefficient"*, ρ. This coefficient is defined for any two non-constant RVs as follows. Fix two random variables X, Y, the correlation coefficient $\rho(X, Y)$ is given by

$$\rho(X, Y) = \frac{\mathbf{Cov}(X, Y)}{\sqrt{\mathbf{V}(X)\mathbf{V}(Y)}} = \frac{\mathbf{Cov}(X, Y)}{\sigma_X \sigma_Y}.$$

The Pearson correlation coefficient is defined only if both of the standard deviations are finite and nonzero. One simple property, derived from the Cauchy–Schwarz inequality, is that $|\rho(X, Y)| \leq 1$. In addition, since the covariance is symmetric, it follows that the correlation coefficient is symmetric. The Pearson correlation coefficient is a kind of measurement for the linear relation between two random variables:

- The Pearson correlation equals $+1$ in the case of a perfect direct (increasing) linear relationship (correlation). That is, in case $Y = aX + b$, when $a, b \in \mathbb{R}$ and $a > 0$.
- The Pearson correlation equals -1 in the case of a perfect decreasing (inverse) linear relationship (anti-correlation). That is, in case $Y = aX + b$, when $a, b \in \mathbb{R}$ and $a < 0$.
- The Pearson correlation equals some value between -1 and $+1$ in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero there is less of a relationship (closer to uncorrelated). The closer the coefficient is to either -1 or 1, the stronger the correlation between the variables.

If the variables are independent, Pearson's correlation coefficient is 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables. For example, suppose the random variable X is symmetrically distributed about zero, and $Y = X^2$. Then Y is completely determined by X, but their correlation is zero, meaning they are uncorrelated.

EXERCISE 12.11. There are two random variables X, Y with a joint probability distribution given in the following table.

$\Pr(X = k, Y = l)$	$l = 6$	$l = 8$	$l = 10$
$k = 1$	0.2	0	0.2
$k = 2$	0	0.2	0
$k = 3$	0.2	0	0.2

Determine whether the RVs are independent? Correlated? Are these results consistent?

Solution. The RVs are dependent since $\Pr(X=1)\Pr(Y=8)>0=\Pr(X=1,Y=8)$. However, they are uncorrelated as

$$\begin{aligned}
\mathbf{E}[X] &= 0.4\cdot 1+0.2\cdot 2+0.4\cdot 3=2,\\
\mathbf{E}[Y] &= 0.4\cdot 6+0.2\cdot 8+0.4\cdot 10=8,\\
\mathbf{E}[XY] &= 0.2\,(6+10+16+18+30)=0.2\cdot 80=16\\
&\Downarrow\\
\mathbf{Cov}(X,Y) &= \mathbf{E}[XY]-\mathbf{E}[X]\,\mathbf{E}[Y]\\
&= 16-16=0.
\end{aligned}$$

The two result are consistent, as a Pearson's correlation coefficient of 0, does not imply independence.

Exercise 12.12. Assume you got 5 cards in your hands, number from 1 to 5, in a random order. Let X be the number of the top card and Y be the number of the bottom one.

(1) Find the correlation coefficient between X and Y.
(2) Let $W=X+Y$. Compute the correlation coefficient between X and W.

Solution. We start with the first question. By symmetry both RVs have the same distribution, namely a uniform distribution such that $X\sim Y\sim U[1,5]$. Also, using symmetry, we can describe the joint distribution of these RVs as follows

$X\setminus Y$	$Y=1$	$Y=2$	$Y=3$	$Y=4$	$Y=5$	Marginal of X
$X=1$	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
$X=2$	$\frac{1}{20}$	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
$X=3$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{5}$
$X=4$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{20}$	$\frac{1}{5}$
$X=5$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	$\frac{1}{20}$	0	$\frac{1}{5}$
Marginal of Y	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	

Clearly, the probability $\Pr(X=k,Y=k)$ equals zero as the tom card and bottom card cannot be the same. And, e.g., the probability of $\Pr(X=1,Y=2)$ is computed as follows:

$$\begin{aligned}
\Pr(X=1,Y=2) &= \Pr(Y=2|X=1)\Pr(X=1)\\
&= \frac{1}{4}\cdot\frac{1}{5}=\frac{1}{20},
\end{aligned}$$

when the second line holds since the probability of $Y=2$ when the top card is one, is 0.25, by symmetry. In order to compute $\rho(X,Y)$ we need to find the variances of both RVs and the covariance. The variance is given by the know formula

$$\begin{aligned}
\frac{(b-a+1)^2-1}{12} &= \frac{(5-1+1)^2-1}{12}\\
&= \frac{24}{12}\\
&= 2=\mathbf{Var}(X)=\mathbf{Var}(Y).
\end{aligned}$$

The covariance could be computed directly.

$$\begin{aligned}
\mathbf{E}[XY] &= [2+3+4+5+6+8+10+12+15+20]\cdot\frac{1}{10}=8.5,\\
\mathbf{E}[X]=\mathbf{E}[Y] &= \frac{5+1}{2}=3,\\
\Rightarrow\ \mathbf{Cov}(X,Y) &= 8.5-3\cdot 3=-0.5.
\end{aligned}$$

Thus,

$$\rho(X,Y)=\frac{-0.5}{\sqrt{2\cdot 2}}=-0.25.$$

Now we can use this result to answer the second question.

$$\begin{aligned}\mathbf{Cov}(X,W) &= \mathbf{Cov}(X,X+Y)\\ &= \mathbf{Cov}(X,X)+\mathbf{Cov}(X,Y)\\ &= \mathbf{V}(X)+\mathbf{Cov}(X,Y)\\ &= 2-0.5=1.5.\end{aligned}$$

In addition,

$$\begin{aligned}\mathbf{V}(W) &= \mathbf{V}(X+Y)\\ &= \mathbf{V}(X)+\mathbf{V}(Y)+2\mathbf{Cov}(X,Y)\\ &= 2+2+2(-0.5)\\ &= 4-1=3.\end{aligned}$$

Thus,

$$\begin{aligned}\rho(X,W) &= \frac{\mathbf{Cov}(X,W)}{\sqrt{\mathbf{V}(X)\mathbf{V}(W)}}\\ &= \frac{\frac{3}{2}}{\sqrt{2\cdot 3}}=\frac{1}{2}\cdot\sqrt{\frac{3}{2}}.\end{aligned}$$

EXERCISE 12.13. There are two random variables X, Y with a joint probability distribution given in the following table.

$\Pr(X=k, Y=l)$	$l=10$	$l=20$	$l=30$
$k=1$	0.04	0	0.20
$k=2$	0.07	0	0.18
$k=3$	0.02	0.11	0.07
$k=4$	0.01	0.12	0.18

(1) Construct a table of the joint CDF.
(2) Find the marginals of both RVs.
(3) Find the conditional distribution $\Pr_{X|Y}(k|20)$.
(4) Find the mean of Y and the conditional mean of X given $Y=20$.
(5) Are these RVS independent?
(6) Verify the law of iterated expectation for the mean of X.

Solution.

(1) The joint CDF could be computed directly by summing up the relevant cells in the previous table. Thus,

$\Pr(X=k, Y=l)$	$l=10$	$l=20$	$l=30$
$k=1$	0.04	0.04	0.24
$k=2$	0.11	0.11	0.49
$k=3$	0.13	0.24	0.69
$k=4$	0.14	0.37	1.00

(2) The marginals are:

$$\Pr(X=k)=\begin{cases}0.24, & k=1,\\ 0.25, & k=2,\\ 0.20, & k=3,\\ 0.31, & k=4.\end{cases}$$

$$\Pr(Y=l)=\begin{cases}0.14, & l=10,\\ 0.23, & l=20,\\ 0.63, & l=30.\end{cases}$$

(3) We use the definition of conditional probability in order to compute $\Pr_{X|Y}(k|20)$.

$$\begin{aligned}\Pr_{X|Y}(k|20) &= \Pr(X=k|Y=20)\\ &= \frac{\Pr(X=k,Y=20)}{\Pr(Y=20)}\\ &= \begin{cases}\frac{0}{0.23}, & k=1,\\ \frac{0}{0.23}, & k=2,\\ \frac{0.11}{0.23}, & k=3,\\ \frac{0.12}{0.23}, & k=4,\end{cases}\\ &= \begin{cases}0, & k=1,\\ 0, & k=2,\\ \frac{11}{23}, & k=3,\\ \frac{12}{23}, & k=4.\end{cases}\end{aligned}$$

(4) Using the marginal of Y we get

$$\mathbf{E}[Y]=0.14\cdot 10+0.23\cdot 20+0.63\cdot 30=24.9.$$

Using the previous question we get

$$\begin{aligned}\mathbf{E}[X|Y=20] &= 0\cdot 1+0\cdot 2+\frac{11}{23}\cdot 3+\frac{12}{23}\cdot 4\\ &= \frac{81}{23}.\end{aligned}$$

(5) No, they are dependent as

$$\Pr(X=2,Y=20)=0\neq 0.25\cdot 0.23=\Pr(X=2)\cdot\Pr(Y=20).$$

(6) First we compute the mean of X directly.

$$\mathbf{E}[X]=1\cdot 0.24+2\cdot 0.25+3\cdot 0.20+4\cdot 0.31=2.58.$$

Now we need to compute $\mathbf{E}[X|Y=l]$ where $l=10,20,30$.

$$\begin{aligned}\mathbf{E}[X|Y=10] &= \frac{1\cdot 0.04+2\cdot 0.07+3\cdot 0.02+4\cdot 0.01}{0.14}=2,\\ \mathbf{E}[X|Y=20] &= \frac{1\cdot 0+2\cdot 0+3\cdot 0.11+4\cdot 0.12}{0.23}=3.52,\\ \mathbf{E}[X|Y=30] &= \frac{1\cdot 0.2+2\cdot 0.18+3\cdot 0.07+4\cdot 0.18}{0.63}=2.36,\end{aligned}$$

hence

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X|Y]]\\ &= 0.14\cdot 2+0.23\cdot 3.52+0.63\cdot 2.36=2.58,\end{aligned}$$

and it works.

EXERCISE 12.14. There are two random variables X,Y with a joint probability distribution given in the following table.

$\Pr(X=k,Y=l)$	$l=3$	$l=8$	$l=10$
$k=1$	0.03	0.02	0.20
$k=2$	0.02	0.12	0.05
$k=3$	0.05	0.01	0.21
$k=4$	0.07	0.11	0.11

(1) Find the marginals of both RVs.
(2) Find the conditional distribution $Y|X=3$.
(3) Find the means and variances of both RVs.
(4) Find the covariance $\mathbf{Cov}(X,Y)$.
(5) Find the correlation coefficient of both RVs.
(6) Use the law of iterated expectation to compute the mean of X.

Solution.

(1) The marginals are

$$\Pr(X=k)=\begin{cases}0.25, & k=1,\\ 0.19, & k=2,\\ 0.27, & k=3,\\ 0.29, & k=4.\end{cases}$$

$$\Pr(Y=l)=\begin{cases}0.17, & k=3,\\ 0.26, & k=8,\\ 0.57, & k=10.\end{cases}$$

(2) The conditional distribution is

$$\begin{aligned}\Pr(Y=l|X=3) &= \frac{\Pr(X=3,Y=l)}{\Pr(X=3)}\\ &= \begin{cases}\frac{0.05}{0.27} & l=3,\\ \frac{0.01}{0.27} & l=8,\\ \frac{0.21}{0.27} & l=10.\end{cases}\end{aligned}$$

(3) The means are $\mathbf{E}[X]=2.6$, $\mathbf{E}[Y]=8.29$. The variances are $\mathbf{V}(X)=1.32$, $\mathbf{V}(Y)=6.45$.
(4) The covariance is $\mathbf{Cov}(X,Y)=-0.514$.
(5) The correlation coefficient is -0.176.
(6) We need to compute $\mathbf{E}[X|Y=l]$ where $l=3,8,10$.

$$\begin{aligned}\mathbf{E}[X|Y=3] &= \frac{1\cdot 0.03+2\cdot 0.02+3\cdot 0.05+4\cdot 0.07}{0.17}=2.94,\\ \mathbf{E}[X|Y=8] &= \frac{1\cdot 0.02+2\cdot 0.12+3\cdot 0.01+4\cdot 0.11}{0.26}=2.81,\\ \mathbf{E}[X|Y=10] &= \frac{1\cdot 0.2+2\cdot 0.05+3\cdot 0.21+4\cdot 0.11}{0.57}=2.40,\end{aligned}$$

hence

$$\begin{aligned}\mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X|Y]]\\ &= 0.17\cdot 2.94+0.26\cdot 2.81+0.57\cdot 2.4=2.6,\end{aligned}$$

and it works.

EXERCISE 12.15. Let $F_X(t)$ be the uniform distribution on $[a,b]$ and let $c\in(a,b)$. Show that $F_X(t|X\le c)$ is the uniform distribution on $[a,c]$.

Solution. We can use the definition of conditional distribution such that

$$\begin{aligned}
F_X(t|X \le c) &= \Pr(X \le t|X \le c) \\
&= \frac{\Pr(X \le t, X \le c)}{\Pr(X \le c)} \\
&= \begin{cases} 0, & t < a, \\ \frac{\Pr(X \le t)}{\Pr(X \le c)}, & a \le t \le c, \\ \frac{\Pr(X \le c)}{\Pr(X \le c)} & t > c, \end{cases} \\
&= \begin{cases} 0, & t < a, \\ \frac{\frac{t-a}{b-a}}{\frac{c-a}{b-a}} = \frac{t-a}{c-a}, & a \le t \le c, \\ 1 & t > c, \end{cases}
\end{aligned}$$

and we got a uniform distribution on $[a, c]$.

Exercise 12.16. Consider the tables of probabilities

$\Pr(X = k, Y = l)$	$l = 10$	$l = 20$
$k = -1$	0.1	a
$k = +1$	0.3	b

What values a, b must take such that X, Y are independent?

Solution. In order for these RVs to be independent, we need to make sure that

$$\Pr(X = k, Y = l) = \Pr(X = k)\Pr(Y = l)$$

for every $k = \pm 1$ and for every $l = 10, 20$. Therefore, we get the following equalities.

$$\begin{aligned}
\Pr(X = -1, Y = 10) &= \Pr(X = -1)\Pr(Y = 10) \\
0.1 &= (0.1 + a) \cdot 0.4.
\end{aligned}$$

$$\begin{aligned}
\Pr(X = +1, Y = 10) &= \Pr(X = +1)\Pr(Y = 10) \\
0.3 &= (0.3 + b) \cdot 0.4.
\end{aligned}$$

$$\begin{aligned}
\Pr(X = -1, Y = 20) &= \Pr(X = -1)\Pr(Y = 20) \\
a &= (0.1 + a) \cdot (a + b).
\end{aligned}$$

$$\begin{aligned}
\Pr(X = +1, Y = 20) &= \Pr(X = +1)\Pr(Y = 20) \\
b &= (0.3 + b) \cdot (a + b).
\end{aligned}$$

Which implies that

$$\begin{aligned}
0.25 &= 0.1 + a, \\
0.75 &= 0.3 + b, \\
a &= 0.15, \\
b &= 0.45.
\end{aligned}$$

We can verify that the other two equalities hold given these values. Note that the sum of probabilities is 1.

CHAPTER 13

The Central Limit Theorem (CLT)

The central limit theorem (CLT) is probably the most popular and commonly-used theorem in probability theory. Its strength and significance comes from the fact that we can make very few assumptions on a set of RVs, and still get a very good approximation of their average.

13.1. Laws of large numbers

Before we discuss the CLT, we start we a simpler theorem, called *the law of large numbers.*
Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed (i.i.d.) RVs. We sample each random variable X_n, and get a number x_n. Denote by $\bar{x_n}$ the average of the first n samples. Assume that the expected value of every RV is μ. Since the RVs are i.i.d., the expected value of the average n RVs is also μ. The basic law of large numbers states that $\bar{x_n} \to \mu$ as $n \to \infty$ with probability 1.

THEOREM 13.1. **(The Weak Law of large numbers)** *For every $\epsilon > 0$,*

$$\lim_{n\to\infty} \Pr\left(|\bar{x_n} - \mu| < \epsilon\right) = 1.$$

The weak law states that no matter how small ϵ is, eventually the average sample is close toμ by no more than ϵ. Basically, we see that, independently of the distribution of the RVs, their average converges to the expectation.

13.2. Central Limit Theorem

The CLT improves the result of the weak law of large numbers, in the sense that it tells us how the average of the RVs is distributed.
First, we define the standard normal distribution. Let $Z \sim N(0,1)$ be a normally distributed RV, with mean $\mu = 0$ and standard deviation $\sigma = 1$. Denote its CDF by Φ. That is, for every $t \in \mathbb{R}$,

$$\Pr(Z \leq t) = \Phi(t).$$

For every set $\{X_n\}_{n\in\mathbb{N}}$ of i.i.d. RVs with finite expectation $\mathbf{E}[X_n] = \mu$ and finite variance $\mathbf{V}(X_n) = \sigma^2 > 0$, define the standardized RV Z_n as $Z_n := \frac{\bar{X_n}-\mu}{\sigma/\sqrt{n}}$ when $X_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

THEOREM 13.2. **(The Central Limit Theorem)** *If $\{X_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d. RVs with finite expectation $\mathbf{E}[X_n] = \mu$ and finite variance $\mathbf{V}(X_n) = \sigma^2$, then*

$$\lim_{n\to\infty} \Pr(Z_n \leq t) = \Phi(t).$$

In words, the distribution of the standardized RV converges to the standard normal distribution. Or, equivalently, $Z_n \sim Z$ as $n \to \infty$. Note that this result is also independent of the distribution of X_n.

REMARK 13.1. Usually, we cannot make an infinite number of samples. Therefore, we use the CLT's approximation when $n \geq 30$. When the number of samples is at least 30, the approximation is sufficiently accurate.

EXERCISE 13.1. A financial tool gives any investor the following returns (per day) for each dollar invested

$$X_i = \begin{cases} 10, & \text{w.p. } 0.4, \\ 0.1, & \text{w.p. } 0.6. \end{cases}$$

That is, after a day, the investor either gets back \$10 for every dollar invested (w.p. 0.4), or just 10 cents for every dollar invested (w.p. 0.6). John invests \$10,000. What is the probability that after 150 days he will have more than \$1?

Solution. Denote by X the amount of money John has after 150 days. Let X_i be the return on day $i = 1, \ldots, 150$. Thus,

$$X = 10^4 \prod_{i=1}^{150} X_i.$$

Taking $\log_{10}(\cdot)$ on both sides gives

$$\begin{aligned}
\log_{10}(X) &= \log_{10}\left(10^4 \prod_{i=1}^{150} X_i\right) \\
&= \log_{10}\left(10^4\right) + \log_{10}\left(\prod_{i=1}^{150} X_i\right) \\
&= 4 + \sum_{i=1}^{150} \log_{10}(X_i).
\end{aligned}$$

Computing the expected value and variance of $\log_{10}(X_i)$ yields

$$\begin{aligned}
\mathbf{E}\left[\log_{10}(X_i)\right] &= 1 \cdot 0.4 + (-1) \cdot 0.6 = -0.2, \\
\mathbf{E}\left[\log_{10}^2(X_i)\right] &= 1 \cdot 0.4 + 1 \cdot 0.6 = 1, \\
\mathbf{Var}\left[\log_{10}(X_i)\right] &= \mathbf{E}\left[\log_{10}^2(X_i)\right] - \left(\mathbf{E}\left[\log_{10}(X_i)\right]\right)^2 = 0.96.
\end{aligned}$$

Thus,

$$\begin{aligned}
\Pr(X > 1) &= \Pr\left(\log_{10}(X) > \log_{10}(1)\right) \\
&= \Pr\left(4 + \sum_{i=1}^{150} \log_{10}(X_i) > 0\right) \\
&= \Pr\left(\sum_{i=1}^{150} \log_{10}(X_i) > -4\right) \\
&= \Pr\left(\frac{1}{150}\sum_{i=1}^{150} \log_{10}(X_i) - (-0.2) > -\frac{4}{150} - (-0.2)\right) \\
&= \Pr\left(\frac{\frac{1}{150}\sum_{i=1}^{150} \log_{10}(X_i) + 0.2}{\frac{\sqrt{0.96}}{\sqrt{150}}} > \frac{-\frac{2}{75} + 0.2}{\frac{\sqrt{0.96}}{\sqrt{150}}}\right) \\
&\approx \Pr\left(Z_{150} > 2.17\right) \\
&= 1 - \Phi(2.17) = 0.015.
\end{aligned}$$

Exercise 13.2. On a roulette in a casino there are 38 numbers: 18 reds, 18 blacks, and 2 greens. You can only bet \$1 on either back or red, and in case you choose correctly (a color is randomly drawn), you get an additional dollar back. What is the numbers of rounds needed such that the casino wins (positive gains) with probability of at least 0.95?

Solution. Denote the number of rounds by n. Let X_i be the profit of the casino in round i. We wish to find n such that $\Pr\left(\sum_{i=1}^{n} X_i > 0\right) \geq 0.95$. Note that

$$X_i = \begin{cases} -1, & \text{w.p. } \frac{9}{19}, \\ 1, & \text{w.p. } \frac{10}{19}. \end{cases}$$

Thus, $\mathbf{E}[X_i] = -1 \cdot \frac{9}{19} + 1 \cdot \frac{10}{19} = \frac{1}{19}$, and

$$\begin{aligned} \mathbf{Var}(X_i) &= \mathbf{E}\left[X_i^2\right] - (\mathbf{E}[X_i])^2 \\ &= 1 - \frac{1}{19^2} = \frac{360}{361}. \end{aligned}$$

The CLT yields

$$\begin{aligned} \Pr\left(\sum_{i=1}^{n} X_i > 0\right) &= \Pr\left(\frac{1}{n}\sum_{i=1}^{n} X_i > \frac{1}{n} \cdot 0\right) \\ &= \Pr\left(\bar{X}_n - \frac{1}{19} > 0 - \frac{1}{19}\right) \\ &= \Pr\left(\frac{\bar{X}_n - \frac{1}{19}}{\sqrt{\frac{360}{361}}/\sqrt{n}} > \frac{-\frac{1}{19}}{\sqrt{\frac{360}{361}}/\sqrt{n}}\right) \\ &\approx \Pr\left(Z_n > -\frac{\sqrt{n}}{6\sqrt{10}}\right) \\ &= 1 - \Phi\left(-\frac{\sqrt{n}}{6\sqrt{10}}\right). \end{aligned}$$

We wish that the last term will be greater than 0.95, or equivalently,

$$\begin{aligned} 1 - \Phi\left(-\frac{\sqrt{n}}{6\sqrt{10}}\right) &\geq 0.95 \\ &\Updownarrow \\ 1 - \left(1 - \Phi\left(\frac{\sqrt{n}}{6\sqrt{10}}\right)\right) &\geq 0.95 \\ &\Updownarrow \\ \Phi\left(\frac{\sqrt{n}}{6\sqrt{10}}\right) &\geq 0.95 \end{aligned}$$

Using the normal distribution table, we can see that this holds if

$$\begin{aligned} \frac{\sqrt{n}}{6\sqrt{10}} &\geq 1.645 \\ n &\geq 593. \end{aligned}$$