Comparison of Oracles: Part II*

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Abstract

This paper studies incomplete-information games in which an information provider, an oracle, publicly discloses information to the players. One oracle is said to dominate another if, in every game, it can replicate the equilibrium outcomes induced by the latter. The companion Part I characterizes dominance under deterministic signaling and under stochastic signaling with a unique common knowledge component. The present paper extends the analysis to general environments and provides a characterization of equivalence (mutual dominance) among oracles. To this end, we develop a theory of information loops, thereby extending the seminal work of Blackwell (1951) to strategic environments and Aumann (1976)'s theory of common knowledge.

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1 Introduction

In settings with incomplete information, whether in peace negotiations, business decisions, or financial markets, players lack full knowledge of all factors that influence the outcomes of their decisions. To address such environments, specialized information providers (e.g., peace mediators, business consultants, and rating agencies) operate as neutral *oracles*, selectively disclosing relevant information that can alter strategic behavior and equilibrium outcomes. This paper studies the role of such oracles in games of incomplete information, modeling them as agents who transmit information through general signaling functions to the players.

Our primary objective is to characterize when one oracle *dominates* another and when two oracles are *equivalent*. To this end, we define a partial order of dominance: one oracle dominates another if, in every game, the information structure of the former can induce the same set of equilibrium outcomes as the latter. Naturally, oracles are equivalent under mutual dominance.[1]

Building on Aumann (1976), the notion of a *common knowledge component* (CKC), i.e., the inclusion-wise smallest set that all players can agree upon, plays a central role in our analysis. The intuition for this is rather clear. In an incomplete information game, the relevant set of states for strategic consideration is the corresponding CKC, however an oracle's knowledge is not confined to it. Oracles, who typically possess information that the players do not, cannot always distinguish between states located in different CKCs. Thus, the structure of CKCs governs the interplay between the players' subjective knowledge and the oracle's informational limitations.

The CKC also defines the boundary between the companion Part I (i.e., Lagziel et al., 2025) and the present paper.[2] Specifically, Part I characterizes dominance when oracles are restricted to deterministic signaling functions, and when stochastic signals are permitted but the state space features a unique CKC. Here, we extend the analysis to environments with multiple CKCs and, in addition, provide a general characterization of equivalence.

[1]Note that we abstract away from cases in which the oracle has preferences over players' action profiles or derives utility from their strategic interaction. In this sense, we adopt Blackwell's approach (see Blackwell, 1951), which focuses on comparing signaling structures (namely, experiments) in decision problems, independently of the sender's objectives.

[2]Throughout the paper, we sometimes refer to Lagziel et al. (2025) as "Part I".

Using the structure of multiple CKCs, we introduce the concept of an information loop, the second key element in our characterization. To formally define these loops and present the main results of the current study, we first partition the state space into distinct CKCs. An information loop is then defined as a closed path of states that connects different CKCs through elements of an oracle's partition.

For example, assume there are 4 states $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and two players whose private information is given by the following partitions: $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}\}$. The players' private information induces two CKCs: $C_1 = \{\omega_1, \omega_2\}$ and $C_2 = \{\omega_3, \omega_4\}$. That is, the two players can agree on each of these two events. See the illustration in Figure 1. If the oracle's information is given by the partition $F_1 = \{\{\omega_1, \omega_3\}, \{\omega_2, \omega_4\}\}$, we say that a loop exists, as the different partition elements of F_1 form a closed path between the two CKCs. Namely, $\omega_1 \in C_1$ and $\omega_3 \in C_2$ are joined by a partition element of F_1 and the same holds for $\omega_2 \in C_1$ and $\omega_4 \in C_2$. This yields a sequence of states that starts in C_1, transitions to C_2, and reverts back again to C_1, through different states that serve as entry and exit points from each CKC.

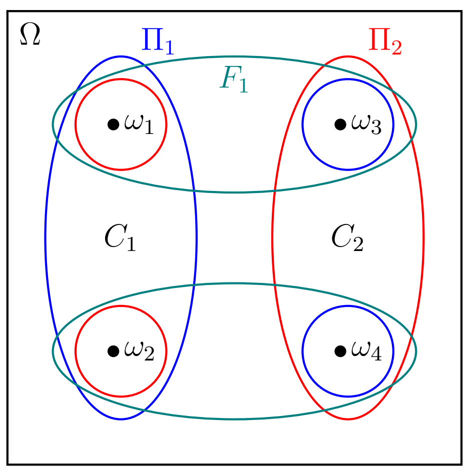


Figure 1: There are two CKCs $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$. The oracle's partition F_1 generates a loop $(\omega_1, \omega_3, \omega_4, \omega_2)$, which is a closed path connecting the two CKCs using the oracle's partition elements.

Assuming that an oracle does not generate information loops, which includes the case where the entire state space comprises a unique CKC, we prove that it dominates the other oracle if and only if its partition refines that of the other within every CKC (see Theorem 1 in Section 3). Importantly, this result extends the characterization result of Part I given a unique CKC, while the refinement condition does not follow from the criterion used in the deterministic setting.

However, if a loop exists, the characterization becomes more complex. An information loop imposes (measurability) constraints on the information the oracle can convey. In the previous example, notice that every signaling function of the oracle over $\{\omega_1, \omega_2\}$ uniquely defines the signaling over $\{\omega_3, \omega_4\}$. Thus, the oracle is not free to signal any information it wants in one CKC, without restricting its ability to convey different information in the other CKC.

An obvious question that goes to the heart of information loops and our results is, why should we care specifically about the signaling structure over the *pairs of states* that form the loop in every CKC? Moreover, why should a loop consist of separate entry and exit points in every CKC? The answer is that, given a CKC, Bayesian updating depends on the ratio of signal-probabilities for the different states. Thus, an effective constraint imposes restrictions over such ratios, thus relating to at least two states in every CKC (while keeping in mind the refinement condition in every CKC; this is a crucial aspect in Lagziel and Lehrer, 2025).

The concept of information loops hints at a significant connection to Aumann's theory of common knowledge, from Aumann (1976). This link appears to be central to understanding how shared and differing information structures impact equilibrium outcomes in incomplete-information games. For this reason we provide an extensive set of results concerning various properties of information loops in Section 4.

Specifically, the first property of information loops that we study is *non-informativeness*. A loop is called *non-informative* if, in every CKC that it intersects, all the states of the loop are in the same partition element of that oracle. We refer to this as non-informativeness because, conditional on the CKC and loop, the oracle has no information to convey to the players. For example, in Figure 1, consider an oracle with a trivial partition $F_1' = \Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. This partition creates a closed path between the two CKCs, as well as joining all the states of the loop (given a CKC) to a single partition element of F_1'. Building on this notion and assuming that the partition of Oracle 1 refines that of Oracle 2 in every CKC, as in the previously stated characterization, then non-informative loops do not pose a problem for dominance and Oracle 1 dominates the other (see Theorem 3 in Section 5).

However, once a loop is *informative* (i.e., in at least one CKC that it intersects, there are

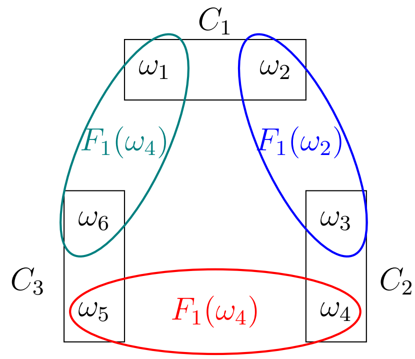


Figure 2: An illustration of a fully-informative and irreducible loop, which intersects three CKCs C_1, C_2 and C_3 with two states in each.

states in the loop from different partition elements of the oracle; see Figure 2), then we require additional conditions for characterization. More specifically, in case there are only two CKCs, an additional condition is that Oracle 2 also has information loops whose states cover Oracle 1's loop, roughly stating the up to non-informative set of states, Oracle 2 has similar loops to those of Oracle 1 (the notion of a cover is formally defined in Section 4). Using this condition we provide a characterization for the case of two CKCs (see Proposition 3 in Section 5.1). While the question of characterization in the case of more than two CKCs remains open, we do provide necessary conditions for dominance in the general case in Theorem 2, building on the notion of *irreducibility*.

The notion of irreducibility, which proves crucial for our analysis, splits to two levels. The first is *irreducible loops*, which implies that there exists no (smaller) loop that is based on a strict subset of states taken from the original loop. The second is referred to as *type-2 irreducible loops*, and it implies that the loop does not contain four states from the same partition element of the oracle (again see Figure 2). On the one hand, type-2 irreducibility is a weaker notion compared to irreducible loops, because it allows for a loop to intersect the same CKC several times, whereas an irreducible loop cannot. On the other hand, a type-2 irreducible loop must be informative because it does not allow for the entry and exit point in every CKC to be in the same partition element of that oracle. In fact, it is *fully-informative* because this condition holds in every CKC, rather than in a specific CKC.

The somewhat delicate understanding of the relations between these loops properties allows us to achieve another main result: the characterization of equivalent oracles. Formally, we say that two oracles are equivalent if they simultaneously dominate one another. The characterization of equivalence, given in Theorem 4 in Section 6, is based on: (i) equivalence in every CKC; (ii) equivalence of irreducible-informative loops; and (iii) a cover over loops. To prove this result, we use type-2 irreducible loops to compare the information of both oracles. Specifically, we consider the sets of type-2 irreducible loops that intersect a joint CKC (i.e., *connected loops*), also taking into account sequential intersections (i.e., the transitive closure) where loop 1 is connected to loop 2 which is then connected to loop 3 and so on. We observe the set of CKCs for each of these groups and refer to these sets as *clusters*. These are used as building blocks in our analysis, and we prove that the information of equivalent oracles must match on these clusters. This, in turn, provides some insight into the possible future characterization of general dominance between oracles, as well as provides another level of extending the theory of common knowledge, beyond information loops.

1.1 Relation to literature

Part II takes the comparison of oracles beyond the two benchmark environments handled in Part I (that is, beyond deterministic signaling and stochastic signaling on a state space with a single CKC), and develops tools for general stochastic signaling when multiple CKCs interact. The central contribution is the introduction of information loops and associated notions: balance, covers, irreducibility (including type-2 irreducibility), and cluster-based aggregation, which together deliver necessary and sufficient conditions in the presence of loops, and a full equivalence characterization that builds on order-preserving covers of irreducible, fully-informative loops.

Our starting point remains Blackwell's comparison of experiments (see Blackwell, 1951, 1953), but the object of comparison and the criterion differ in two key ways. First, an oracle is an *experiment generator*, namely, it can implement any public experiment measurable with respect to its partition, rather than being a fixed experiment. Second, the criterion is *strategic and multi-player*, so dominance is defined by equality of the sets of Nash-equilibrium outcome distributions across all games, holding players' private partitions fixed. These differences matter only weakly with a single CKC, but are crucial with multiple CKCs, where the loop calculus captures exactly how measurability forces cross-component co-movement of posteriors.

Our CKC-based analysis traces back to the epistemic foundations of games, interacting specifically with the common knowledge ideas of Aumann (1976). For Part II, where the state space decomposes into multiple CKCs, the right lens is the approximation of common knowledge by common beliefs *à la* Monderer and Samet (1989), who formalize p-belief and common p-belief, showing how implications that classically require exact common knowledge can be approximated by sufficiently strong common beliefs. The work of Aumann was also followed by Mertens and Zamir (1985), who construct a universal type space embedding all coherent hierarchies of beliefs, thus providing a unified measurable framework for Bayesian games, and by Brandenburger and Dekel (1993), who clarify the equivalence between hierarchies of beliefs and type representations, linking them to common knowledge.

Our model builds on these studies by fixing the partition structures while varying only the oracle's public experiment. The novel constraints we study arise from *global* measurability across CKCs (via loops), not from additional complexity in private belief hierarchies. Our information loops formalize when public measurability (via the oracle's information) stitches distinct CKCs so that posterior ratios must align across components, and when such stitching is slack (no loops) or binding (informative, irreducible loops). This conceptual bridge clarifies why refinement within CKCs suffices absent loops, but not in general.

Relative to information design and persuasion, the present analysis is comparative rather than optimal. The persuasion literature[3] asks which experiment maximizes a sender's objective. Here the oracle has no objective, but is evaluated by its replication ability. In this sense, this project complements persuasion by characterizing when two generators of public experiments are equivalent or when one dominates another.

Closer to us, Kolotilin et al. (2017) analyze persuasion with a privately informed receiver and establish conditions under which optimal mechanisms can be represented as experiments, delivering tractable characterizations in linear/monotone environments. Part II treats the players' experiments as primitives, but evaluates an oracle by the ability to replicate another across all games with fixed private information, so that the binding obstacles are global measurability (loops) rather than incentive constraints.

[3]For a recent survey, see Kamenica (2019).

Another strand in the literature studies mediators in games with incomplete information. Mediators deliver differential recommendations that coordinate players' actions and implement variants of correlated equilibria (Forges, 1993). In many formulations the mediator does not convey additional information about the realized state; i.e., its role is purely coordinative. Under complete information, Gossner (2000) compares mediating structures by the sets of correlated equilibria they can induce, calling one device "richer" if it generates a superset. This characterization uses a notion of compatible interpretation in the spirit of garbling. Part II departs from this strand in two respects: the oracle's messages are public and informational about the state, and comparison is by replication power across all games. With multiple CKCs, feasibility is governed not by recommendation schemes but by measurability links across CKCs, captured in our framework by information loops (balance and covers).

Closer to the present project are studies on incomplete-information games that establish partial orderings of information structures. Peski (2008) obtains a Blackwell-type ordering in zero-sum games. Lehrer et al. (2010) analyze common-interest games with privately observed, possibly correlated signals, showing that comparative results hinge on the version of Blackwell garbling tied to the chosen solution concept. Lehrer et al. (2013) extend garbling to characterize *outcome equivalence*. Bergemann and Morris (2016) study n-player environments via Bayes correlated equilibrium and characterize dominance through *individual sufficiency*. Part II differs along two margins crucial with multiple CKCs: (i) players' private partitions are fixed primitives while the oracle is an experiment generator of public signals; and (ii) dominance/equivalence are defined by the ability to reproduce the set of equilibrium outcome distributions in *every* game, and thus hinge on the loop calculus rather than garbling alone.

The structure of the paper. The paper is organized as follows. Section 2 depicts the model. Section 3 provides a characterization of dominance when there are no loops. Section 4 studies the properties of information loops. Section 4 outlines necessary and sufficient conditions for dominance, as well as a characterization of dominance given two CKCs (in Section 5.1). Finally, in Section 6 we characterize the equivalence relation between oracles.

2 The model

A *guided game* consists of a Bayesian game together with an *oracle*. The oracle provides information intended to enable a different, and preferably broader, set of equilibria. It operates via signaling, and our analysis characterizes the extent to which oracles can expand the set of equilibrium payoffs.

We begin by defining the underlying Bayesian game. Let $N = \{1, 2, \dots, n\}$ be a finite set of $n \geq 2$ players, and let Ω be a non-empty, finite state space. Each player $i \in N$ has a non-empty, finite action set[4] A_i and an information partition Π_i of Ω. Let $A = \times_{i \in N} A_i$ denote the set of action profiles. Player i's utility is $u_i : \Omega \times A \to \mathbb{R}$, mapping states and action profiles to payoffs.

To extend the basic game to a guided game, we introduce an oracle that provides public information before actions are chosen. The oracle has a partition F of Ω and a countable signal set S. A strategy of the oracle is an F-measurable function $\tau : F \to \Delta(S)$ with finite-support distributions, used to transmit information to all players N, where $\Delta(S)$ denotes the set of finite-support probability distributions over S. For $\omega \in \Omega$ and $s \in S$, we write $\tau(s \mid \omega)$ for the probability $\tau(\omega)(s)$ that s is sent when the realized state is ω. Note that any deterministic strategy $\tau : F \to S$ is effectively a partition, and we refer to it as such when appropriate.

The guided game evolves as follows. First, the oracle publicly announces a strategy τ. Then, a state $\omega \in \Omega$ is drawn according to a common prior $\mu \in \Delta(\Omega)$. Each player i is privately informed of $\Pi_i(\omega)$, the atom (i.e., set of states) of player i's partition that contains ω. Finally, if τ is deterministic, the signal $\tau(\omega) \in S$ is publicly announced, and if τ is stochastic, a realization $s \in S$ is drawn according to $\tau(\omega)$ and publicly announced.

Let the join[5] $\Pi_i \vee F'$ denote the updated partition of player i given Π_i and a partition F'. If τ is deterministic, define $\mu^i_{\tau|\omega} = \mu\big(\cdot \mid [\Pi_i \vee \tau](\omega)\big) \in \Delta(\Omega)$ as player i's posterior after observing $\Pi_i(\omega)$ and $\tau(\omega)$. If τ is stochastic, let $\mu^i_{\tau|\omega,s} = \mu(\cdot \mid \Pi_i(\omega), \tau, s) \in \Delta(\Omega)$ denote player i's posterior after observing $\Pi_i(\omega)$ and a realized signal s according to $\tau(\omega)$. Thus, every strategy τ induces an incomplete-information game $G(\tau) = (N, (A_i)_{i \in N}, (\mu^i_\tau)_{i \in N}, (u_i)_{i \in N})$. Since

[4] In this framework, A_i is independent of the player's information, but the setting can also accommodate cases where it is not.

[5] Coarsest common refinement of Π_i and F'; following Aumann (1976).

the state space and action sets are finite, the Nash equilibria exist. When there is no risk of ambiguity, we denote the incomplete-information game without τ by G.

2.1 Partial ordering of oracles

To discuss the oracle's role in this framework, we adopt a solution concept, referred to as a *Guided equilibrium*, that incorporates the oracle's strategy. Let $\sigma_i : \Pi_i \times S \to \Delta(A_i)$ be a strategy for player i. A tuple $(\tau, \sigma_1, \ldots, \sigma_n)$ is a *Guided equilibrium* if $(\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium of the incomplete-information game $G(\tau)$.

This notion of a Guided equilibrium induces a partial order over oracles (that is, over their partitions) via the sets of equilibria they can generate. Let $\mathrm{NED}(G(\tau)) \subseteq \Delta(\Omega \times A)$ denote the set of distributions over $\Omega \times A$ induced by Nash equilibria given G and τ.[6] Now consider two oracles, Oracle 1 and Oracle 2, and let F_j and τ_j denote the partition and strategy of Oracle j, respectively. Using this notation, we define a partial order as follows.

Definition 1 (Partial ordering of Oracles)**.** Oracle 1 dominates Oracle 2*, denoted* $F_1 \succeq_{\mathrm{NE}} F_2$*, if for every* τ_2 *and game* G*, there exists* τ_1 *such that* $\mathrm{NED}(\mathrm{G}(\tau_1)) = \mathrm{NED}(\mathrm{G}(\tau_2))$.

Informally, dominance means that one oracle can replicate the other's signaling structure so as to induce the same set of equilibria. A direct comparison of equilibria across games without conditioning on the signaling rule is problematic because players' strategies typically depend on the oracle's signals.

2.2 More than one CKC: two examples

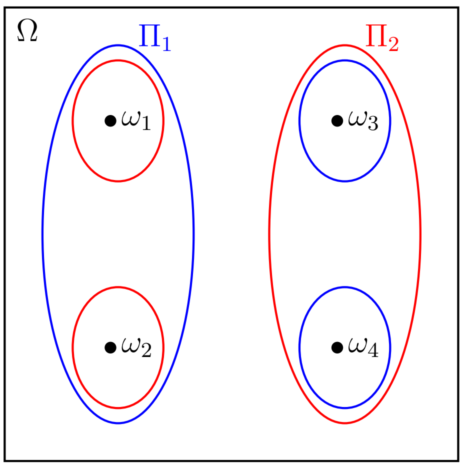
The partition-refinement condition given in Lagziel et al. (2025) ensures that Oracle 1 can produce the *exact* same strategy as Oracle 2. This however, hinges on the existence of a unique CKC. In case there are several CKCs, Oracle 1 may need to follow a different strategy in order to match the distribution on posteriors generated by τ_2. Namely, τ_1 may require more signals

[6] A Nash equilibrium $(\sigma_1^*, \ldots, \sigma_n^*)$, along with the common prior μ, induce a probability distribution on $\Omega \times A$. Fix ω and an action profile a. The probability of (ω, a) under $(\sigma_1^*, \ldots, \sigma_n^*)$, τ and the common prior μ equals $\mu(\omega) \sum_{s \in S} \tau(s \mid \omega) \prod_{i=1}^{n} \sigma_i^*(a_i \mid \Pi_i(\omega), s)$. As multiple equilibria may exist, $\mathrm{NED}(G(\tau))$ is a subset of $\Delta(\Omega \times A)$.

than τ_2, even if both oracles have the same (complete) information in every CKC. Let us provide a concrete example for this.

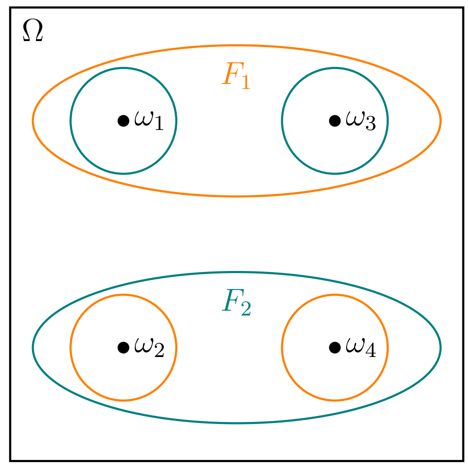
Example 1. *More signals are needed.*

Consider a uniformly distributed state space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, with two players whose private information is $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ and $\Pi_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}\}$. The oracles have the following partitions $F_1 = \{\{\omega_1, \omega_3\}, \{\omega_2\}, \{\omega_4\}\}$ and $F_2 = \{\{\omega_1\}, \{\omega_3\}, \{\omega_2, \omega_4\}\}$. This information structure is illustrated in Figure 3. Notice that there are two CKCs, $\{\omega_1, \omega_2\}$ and $\{\omega_3, \omega_4\}$, and both oracles have complete information in each of these components. That is, F_1 refines F_2 in every CKC, and vice versa.



(a)

The players' information



(b)

The oracles' information

Figure 3: On the left, Figure (a) illustrates the information structure of player 1 (blue) and player 2 (red). On the right, Figure (b) portrays the information structure of Oracle 1 (orange) and Oracle 2 (green).

Consider the stochastic strategy τ_2 given in Figure 4. Notice it is F_2-measurable, as $\tau_2(s|\omega_2) = \tau_2(s|\omega_4)$ for every signal s, but not F_1-measurable.

The set $\text{Post}(\tau_2)$ of τ_2-posteriors is

$$\text{Post}(\tau_2) = \left\{ \begin{array}{ll} (e_i, e_i), & \forall\ 1 \le i \le 4, \\ \left(\left(\frac{3}{7}, \frac{4}{7}, 0, 0\right), e_j\right), & j = 1, 2, \\ \left(e_k, \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)\right), & k = 3, 4 \end{array} \right\},$$

$\tau_2(s\|\omega)$	s_1	s_2	s_3
ω_1	0	1/2	1/2
ω_2	1/3	2/3	0
ω_3	0	2/3	1/3
ω_4	1/3	2/3	0

Figure 4: A stochastic F_2-measurable strategy of Oracle 2.

and we can now try to mimic τ_2 using an F_1-measurable strategy. First, this requires at least two signals to distinguish between ω_1 and ω_2, as well as ω_3 and ω_4. Second, the posterior $\left(\left(\frac{3}{7}, \frac{4}{7}, 0, 0\right), e_1\right)$ requires another signal s so that $\tau(s|\omega_1) = \alpha > 0$ and $\tau(s|\omega_3) = \frac{4}{3}\alpha > 0$. However, the F_1-measurability requirement implies that $\tau(s|\omega_3) = \alpha$, and the τ_2-posterior $\left(e_3, (0, 0, \frac{1}{2}, \frac{1}{2})\right)$ necessitates that $\tau(s|\omega_4) = \alpha$ as well. These conditions are jointly given in Table (a) within Figure 5.

$\tau_1(s\|\omega)$	s_3	s_4	s_5
ω_1	α	β	0
ω_2	$\frac{4}{3}\alpha$	0	γ
ω_3	α	β	0
ω_4	α	0	γ

(a)

$\tau_1(s\|\omega)$	s_3	s_4	s_5	s_6
ω_1	1/2	1/3	0	1/6
ω_2	2/3	0	1/3	0
ω_3	1/2	1/3	0	1/6
ω_4	1/2	0	1/3	1/6

(b)

Figure 5: A strategy τ_1, either with 3 signals as given in Table (a), or with 4 signals as in Table (b).

Evidently, it must be that $\alpha, \beta, \gamma > 0$ in order to mimic τ_2, but the second and fourth rows in Table (a) cannot jointly sum to 1 unless $\alpha = 0$, which eliminates the possibility of a well-defined mimicking strategy. Thus, in order to mimic the stated strategy τ_2, Oracle 1 requires an additional signal as presented in Table (b), in Figure 5. To conclude, though the oracles' partitions refine one another in every CKC, they cannot always produce the exact same strategy when trying to mimic each other.

Example 2. *Dominance need not imply refinement with multiple* CKCs

In this example we wish to show that when there are multiple CKCs, Oracle 1 can dominate Oracle 2 although F_1 does not refine F_2. To see this, we revisit an example from Lagziel et al. (2025) in which $\Pi_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, F_1 = \{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}\}$ and $F_2 = \{\{\omega_1, \omega_2\}, \{\omega_3\}, \{\omega_4\}\}$. This is illustrated in Figure 6.

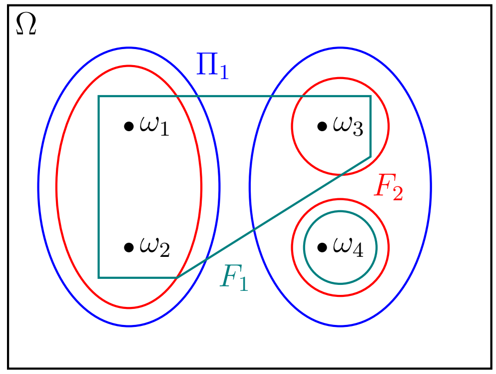


Figure 6: Note that F_2 strictly refines F_1 and Π_1.

Now consider the signaling strategy of Oracle 2 given in Figure 7, where Oracle 2 provides the players with no additional information regarding states ω_1 and ω_2. Thus, the posterior over these states remains the original one. On the other hand, given the states ω_3 and ω_4, the strategy τ_2 reveals the true state with a positive probability and induces the posterior $(0, 0, 2/5, 3/5)$ with the remaining probability.

$\tau_2(s\|\omega)$	s_1	s_2	s_3
ω_1	1/4	0	3/4
ω_2	1/4	0	3/4
ω_3	0	1/2	1/2
ω_4	1/4	0	3/4

Figure 7: A stochastic F_2-measurable strategy of Oracle 2.

While Oracle 2 can assign different probabilities to a signal conditioned on ω_2 and ω_3, Oracle 1 cannot. However, there is a signaling strategy for Oracle 1 that produces the same distribution over the posteriors as τ_2 does. The following strategy τ_1, given in Figure 8, does that.

$\tau_1(s\|\omega)$	s_1	s_2	s_3
ω_1	1/2	0	1/2
ω_2	1/2	0	1/2
ω_3	1/2	0	1/2
ω_4	0	1/4	3/4

Figure 8: A stochastic F_1-measurable strategy of Oracle 1.

In this example, it is straightforward to prove that Oracle 1 can mimic every strategy τ_2 of Oracle 2, and we prove this result under more general conditions in Theorem 1 and Proposition

3. Yet, it is clear that F_1 is not a refinement of F_2 in general, but it is a refinement in every CKC.

3 Multiple CKCs and no loops

We now turn to the general setting in which the players' information structures induce any (finite) number of CKCs. Assume that $C_1, \ldots, C_l$ are mutually exclusive CKCs such that $\Omega = \bigcup_{j=1}^{l} C_j$. A key aspect of our analysis is the presence of measurability constraints, where different CKCs are connected by atoms of the oracles' partitions. To understand the significance of this, consider a setting where F_1 does not contain any element intersecting multiple CKCs. In this case, the characterization result given a unique CKC from Part I (see Theorem 5 in Appendix A.1.2) applies separately to each CKC, as Oracle 1 faces no constraints when attempting to mimic some strategy of Oracle 2.

However, when elements of Oracle 1's partition intersect different CKCs, the analysis becomes more complex, because we must account for measurability constraints when attempting to use the same strategy τ_1 across different CKCs. Such intersections impose constraints on τ_1, preventing us from naively applying previous results.

This issue becomes even more complicated when multiple elements of Oracle 1's partition intersect different CKCs, forming what we call an (information) *loop*.[7]

Generally, a loop is an ordered sequence of states from different CKCs such that the partition of an oracle groups together distinct pairs of states from different CKCs, creating a closed path. The main result of this section, presented in Theorem 1 below, states that in the absence of such loops, Oracle 1 dominates Oracle 2 if and only if F_1 refines F_2 in every CKC. The formal definition of a loop is provided in Definition 2.

Definition 2. *An F_i-loop is a sequence $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, where $m + 1 \equiv 1$ and $m \geq 2$, such that*

- $\omega_j, \overline{\omega}_j \in C_{r_j}$ *and* $\omega_j \neq \overline{\omega}_j$ *for all* $j = 1, \ldots, m$.[8]

[7] An (information) loop is different from a loop in graph theory. In graph theory, a loop refers to an edge that connects a vertex to itself.

[8] Here C_{r_j} refers to the CKC that contains the j-th pair of states $(\omega_j, \overline{\omega}_j)$.

- $\omega_{j+1} \in F_i(\overline{\omega}_j)$ *for all* $j = 1, \ldots, m$.

- $C_{r_j} \neq C_{r_{j+1}}$ *for all* $j = 1, \ldots, m$.

- *The sets* $\{\overline{\omega}_j, \omega_{j+1}\}$ *are pairwise disjoint for all* $j = 1, \ldots, m$.

To understand information loops, one can view the CKCs as the vertices of a graph. An edge connects two CKCs if there exist ω_{j+1} and $\overline{\omega}_j$ such that they belong to the same F_i-partition element (this corresponds to the second requirement). An information loop then parallels an Eulerian graph, where there is a walk that includes every edge exactly once (the last requirement in the definition) and ends back at the initial vertex (hence the requirement $m + 1 \equiv 1$). As noted at the beginning of Section 3, the key aspect of the general analysis is to consider the case when the oracle partition atoms intersect different CKCs, so we require that $C_{r_j} \neq C_{r_{j+1}}$ for all $j = 1, \ldots, m$.

An example of an F_1-loop is provided in Figure 9.(a), which depicts a loop consisting of six states across three CKCs. Note that a loop can intersect the same CKC multiple times, as long as the sets $\{\overline{\omega}_j, \omega_{j+1}\}$ remain pairwise disjoint for each j.

We use the concept of a loop in our first general characterization, presented in Theorem 1. This theorem builds on the assumption that F_1 contains no loops and extends the main result of Part I by showing that one oracle dominates another if the former's partition refines that of the latter in every CKC. It is important to note that the proof is extensive, as it must account for the measurability constraints of τ_1 across all CKCs.

Theorem 1. *Assume there is no* F_1*-loop. Then,* Oracle 1 *dominates* Oracle 2 *if and only if* F_1 *refines* F_2 *in every* CKC.

The proof of Theorem 1 builds on the concept of a *sub-strategy*. A sub-strategy is a signaling function without the requirement that the probabilities sum to 1. This relaxation allows us to study functions that partially mimic a strategy τ_2, meaning each posterior is drawn from $\mathrm{Post}(\tau_2)$ and is induced with a probability that does not exceed the probability with which τ_2 induces it. We show that the set of sub-strategies is compact, allowing us to consider an optimal sub-strategy for mimicking τ_2. The proof then proceeds by contradiction: if the optimal

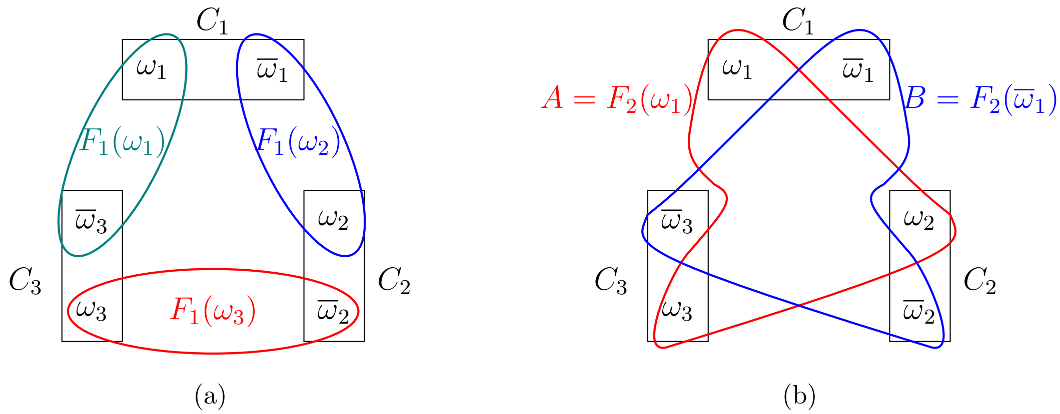


Figure 9: Figure (a) depicts an F_1-loop with three CKCs and six states overall. Figure (b) illustrates how the F_1-loop, presented in (a), is non-balanced with respect to F_2. Namely, F_2 has two elements $A = \{\omega_1, \omega_2, \omega_3\}$, and $B = \{\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3\}$ such that the number of transitions from A to B are 3, while the reverse equals 0.

sub-strategy is not a complete strategy, we can extend it by constructing an additional sub-strategy to complement the optimal one for posteriors that are not fully supported (relative to the probabilities induced by τ_2). This part is rather extensive as it requires some graph theory and several supporting claims given in the proof in the appendix.

4 Information loops

Previous sections have examined the problem of oracle dominance in the absence of loops, considering either a unique CKC or multiple CKCs without loops. However, in order to confront the general question of dominance in the presence of information loops, we need to have a clear understanding of their properties and implications.

Specifically, when an F_1-loop exists, it may create challenges for Oracle 1 in mimicking Oracle 2, because loops introduce measurability constraints across CKCs. Although Oracle 1 can mimic Oracle 2 within each CKC individually, it may be impossible to do so simultaneously across CKCs if the required combined strategy is not measurable with respect to F_1. This suggests that any F_1-loop must satisfy certain conditions to ensure that such a strategy is indeed F_1-measurable. The first condition that we study, which turns out to be a necessary condition for dominance, is generally referred to as F_2-*balanced*.

The idea starts with an F_1-loop. We examine all states in this loop and determine how they can be covered by two F_2-measurable sets. In other words, the loop is divided into two disjoint sets, each contained in an F_2-measurable set, denoted A and B. Next, we count the number of transitions along the loop from A to B, where the entry point into one CKC is through a state in A and the exit is through a state in B. We do the same for transitions from B to A. An F_1-loop is called F_2-*balanced* if the number of transitions between A and B is equal in both directions. The formal definition follows.

Definition 3. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_{-i}-*balanced *if for every F_{-i}-measurable partition of the loop's states into two disjoint sets $\{A, B\}$ such that $\cup_j\{\omega_j, \overline{\omega}_j\} \subseteq A \cup B$, it follows that:*

$$\#(A \to B) := |\{j; \omega_j \in A \text{ and } \overline{\omega}_j \in B\}| = |\{j; \omega_j \in B \text{ and } \overline{\omega}_j \in A\}| =: \#(B \to A). \tag{1}$$

Note that an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, where $\omega_j \in F_2(\overline{\omega}_j)$ for all $j = 1, \ldots, m$, is F_2-*balanced*. Figure 9.(b) examines the F_1-loop from Figure 9.(a). The sets A and B are F_2-measurable, restricted to the six states under consideration. The partition into A and B renders the loop non-F_2-balanced, as $\#(A \to B) = 3$, while $\#(B \to A) = 0$.

Why are balanced loops crucial? Consider, for example, a non-balanced loop as depicted in Figure 9, and assume that $\tau_2(s|\omega) = \frac{1}{2} - \frac{1}{4}\mathbf{1}_{\{\omega \in A\}}$ for some signal $s \in S$. This imposes a specific $1:2$ ratio between any two states described in each CKC, so that $\Pi_i \frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)} = \frac{1}{8}$. However, since $\overline{\omega}_i$ and ω_{i+1} belong to the same F_1 partition element, the measurability constraints on Oracle 1 along the loop require that $\tau_1(s|\overline{\omega}_i) = \tau_1(s|\omega_{i+1})$, hence $\Pi_i \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = 1$ for any s in the support of all states. In other words, Oracle 1 cannot match the ratio dictated by τ_2, therefore the key proportionality lemma from Part I (see Lemma 1 from in Appendix A.1.1) does not hold in at least one CKC.

If the loop were balanced—say, with $A = \{\overline{\omega}_1, \omega_2\}$ and $B = \{\omega_1, \overline{\omega}_2, \omega_3, \overline{\omega}_3\}$—then the same strategy τ_2 would yield $\Pi_i \frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)} = 1$, as required. In general, when all loops are balanced, this discrepancy is eliminated for any two such sets A and B. The notion of balanced loops is closely related to the following notion of *covered loops*, which implies that an F_1-loop can be decomposed to loops of F_2.

Definition 4. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_{-i}-*covered *if*

- *The set $\{1, ..., m\}$ is partitioned to disjoint sets of indices, $J, I_1, ..., I_r$, i.e., $\{1, ..., m\} = J \cup (\cup_{t=1}^{r} I_t)$.*
- *For each $t = 1, ..., r$, $\Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I_t}$ is an F_{-i}-loop, also referred to as a* sub-loop.[9]
- *$J = \{j; \omega_j \in F_{-i}(\overline{\omega}_j)\}$.*

The cover is order-preserving *if every F_{-i}-loop $\Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I_t}$ in the cover follows the same ordering of pairs as the F_i-loop.*

In simple terms, the definition states that, given an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$, we can partition its states to several F_2-loops and a set of states where $\omega_j \in F_2(\overline{\omega}_j)$. Figure 10 (a) depicts an F_1-loop consisting of $((\omega_j, \overline{\omega}_j))_{j=1,...,4}$, which is covered by two F_2-loops: $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3)$ and $(\omega_2, \overline{\omega}_2, \omega_4, \overline{\omega}_4)$. In this case, the set J (defined in Definition 4) is empty. Figure 10 (b) depicts a case in which $J = \{2, 4\}$, and $(\omega_1, \overline{\omega}_1, \overline{\omega}_3, \omega_3)$ forms an F_2-loop, yet it is not an F_2-sub-loop of the original F_1-loop since $\overline{\omega}_1$ is linked to $\overline{\omega}_3$ instead of ω_3. Actually, if we set $A = \{\omega_2, \overline{\omega}_2, \omega_4, \overline{\omega}_4, \omega_1, \omega_3\}$ and $B = \{\overline{\omega}_1, \overline{\omega}_3\}$, which are F_2-measurable, then $\#(A \to B) = 2$, but $\#(B \to A) = 0$, so the F_1-loop is not F_2-balanced. Finally, note that the sub-loops in Figure 10 (a) are order-preserving. By contrast, the sub-loop $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3, \omega_2, \overline{\omega}_2)$ in Figure 10 (c) does not preserve the ordering of the pairs as the F_1-loop, since the pair $(\omega_3, \overline{\omega}_3)$ appears before $(\omega_2, \overline{\omega}_2)$.

The following Proposition 1 proves that an F_1-loop is F_2-balanced if and only if it is F_2-covered. This proposition assists with the proof of Theorem 2 below, which provides a necessary condition for dominance.

Proposition 1. *Let $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ be an F_1-loop. The following statements are equivalent:*

i. *The loop is F_2-balanced;*

[9] The order of the pairs $(\omega_j, \overline{\omega}_j)$ in the F_{-i}-loop does not have to coincide with their order under the F_i-loop. For instance, an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \omega_3, \overline{\omega}_3)$ might be covered by the following F_2-loop $(\omega_1, \overline{\omega}_1, \omega_3, \overline{\omega}_3, \omega_2, \overline{\omega}_2)$.

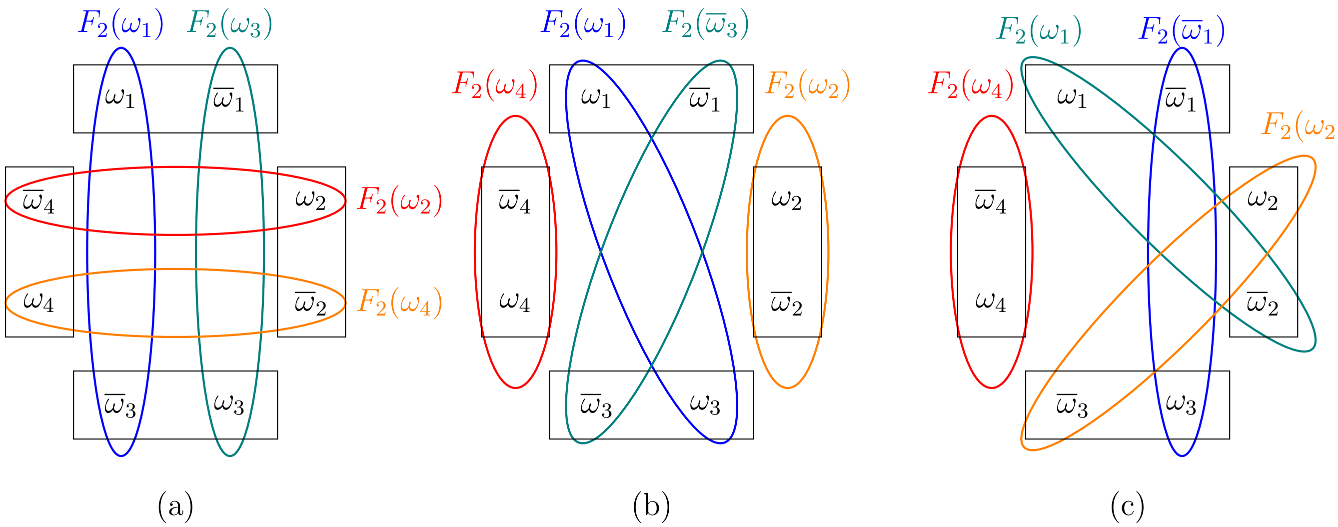


Figure 10: Two states connected by a colored line are in the same information set of F_2. In (a), the F_2-sub-loops that cover the F_1-loop are order-preserving, i.e., following the ordering of pairs in the original F_1-loop, whereas the sub-loop in (c) is not order-preserving. (b) illustrates a case where $(\omega_1, \overline{\omega}_1, \overline{\omega}_3, \omega_3)$ forms an F-2 loop, but it is not an F_2-sub-loop of the original F_1-loop.

ii. *The loop is F_2-covered;*

iii. *For every F_2-measurable function $f : \{\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m\} \to (0, \infty)$,*

$$\prod_{i=1}^{m} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1.$$

The next two properties that we study are *irreducible* and *informative* loops. Starting with the former, an F_i-loop is irreducible if it does not have a *sub-loop*, namely, there exists no 'smaller' F_i-loop that comprises a strictly smaller set of states taken solely from the original loop. Our analysis would use irreducible loops as building blocks to decompose and compare loops generated by the oracles' partitions.

Definition 5. *Let $L_i = (\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ be an F_i-loop. We say that the loop is* irreducible *if there exists no strict subset of the set $\{\omega_j, \overline{\omega}_j : j = 1, \ldots, m\}$ that forms an F_i-loop.*

We use the definition of an irreducible loop in the context of covers as well, stating that a cover is *irreducible* if every loop in the cover is irreducible. Furthermore, the idea of irreducible loops is closely related to the concept of covers, and specifically to the set $J = \{j; \omega_j \in F_{-i}(\overline{\omega}_j)\}$ given in Definition 4 above. Specifically, if there exists an F_i-loop with a pair of states $(\omega_j, \overline{\omega}_j)$ such that $\overline{\omega}_j \in F_i(\omega_j)$, then it cannot be irreducible unless it comprises only 4 states.[10] We typically refer to such cases where $\overline{\omega}_j \in F_i(\omega_j)$ as *non-informative* because Oracle i cannot distinguish between the two states. This condition is essentially equivalent to every F_1-loop being F_2-balanced at 0, meaning that for any choice of the specified F_2-measurable sets A and B, the number of transitions between these sets is zero. The following Definition 6 captures the idea of *informative* loops, which would later be used in Theorem 3 as a sufficient condition for dominance.

Definition 6. *An F_i-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is F_k-*non-informative *if $F_k(\omega_j) = F_k(\overline{\omega}_j)$ for every j. The loop is F_k-*fully-informative *if $F_k(\omega_j) \neq F_k(\overline{\omega}_j)$ for every j.*

To understand the motivation behind this definition, consider any F_1-loop denoted by $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$. If this loop is F_2-non-informative, it suggests that the ratios $\frac{\tau_2(s|\omega_i)}{\tau_2(s|\overline{\omega}_i)}$ equals 1 for every signal s supported on these states. In simple terms, conditional on any $\{\omega_i, \overline{\omega}_i\}$, Oracle 2 does not provide any additional information, so the constraints that an F_1-loop imposes on Oracle 1 in every CKC (i.e., that the product of probability ratios along the loop equals 1) are met by the measurability requirements of F_2.

The following proposition summarizes key properties of informative and irreducible loops. It states that an irreducible loop intersects every CKC at most once and must be fully informative (unless it has only 4 states). In addition, the proposition shows that an informative loop has a fully-informative sub-loop, as well.

Proposition 2. *Consider an F_i-loop L_i.*

- *If L_i intersects the same* CKC *more than once, then it is not irreducible.*
- *If L_i is irreducible and consists of at least* 6 *states, then it is F_i-fully-informative.*
- *If L_i is F_i-informative, then it has an F_i-fully-informative sub-loop.*
- *If L_i is F_i-fully-informative, then it can be decomposed to irreducible F_i-loops.*

[10] In general, the smallest possible loop has at least 4 states, so any such loop is, by definition, irreducible.

- *If L_i is not irreducible, then either it intersects the same* CKC *more than once, or it has at least 4 states in the same partition element of F_i.*

We use this proposition in the following subsection to provide necessary and sufficient conditions for the dominance of one oracle over another.

5 Necessary and Sufficient conditions for dominance

In the following section, we address the general case where F_1 has loops, which imposes constraints on Oracle 1 *across* CKCs. Due to the complexity of this problem, we divide our analysis into two parts: a necessary condition for dominance presented in Theorem 2, and a sufficient condition given in Theorem 3. These theorems depend strongly on the properties of information loops, and specifically on the notions of covers, irreducibility and non-informativeness.

Starting with the necessary conditions, the following theorem, which builds on Propositions 1 and 2, states that if Oracle 1 dominates Oracle 2, then besides the refinement condition in every CKC, already established in Theorem 1, it must be that every F_1-loop is covered by loops of F_2. In addition, it states that every irreducible F_2-loop that cover an irreducible F_1-loop is order-preserving, essentially stating that the two loops coincide.

Theorem 2. *If* Oracle 1 *dominates* Oracle 2*, then:*

- *F_1 refines F_2 in every* CKC*;*
- *Any F_1-loop has a cover by F_2-loops; and*
- *Every irreducible F_2-loop that covers an irreducible F_1-loop is order-preserving.*

The proof of the first part is immediate, as it follows directly from the main result of Part I (see Theorem 4 therein cited in Appendix A.1.2). The proof of the second part relies on Proposition 1 by assuming that an F_1-loop is not F_2-balanced, and constructing a strategy τ_2 that Oracle 1 cannot mimic without violating measurability constraints. The last part relies on Proposition 2, as well as a key lemma from Part I (cited in Appendix A.1.1), by depicting a two-signal strategy τ_2 that one cannot mimic without following the same order of pairs throughout the F_2-loop.

Next, we use the understanding regarding covered and balanced loops to present a sufficient condition for dominance, which indirectly requires that any loop is balanced at 0—meaning that there are no transitions between sets A and B. This leads to the following Theorem 3, which uses the non-informative notion for dominance.

Theorem 3. *If F_1 refines F_2 in every* CKC *and every F_1-loop is F_2-non-informative, then* Oracle 1 *dominates* Oracle 2.

Though we do not yet provide a full characterization, it becomes rather clear that the requirement that every F_1-loop is F_2-balanced should be the main focus, as it is a necessary condition, as well as a sufficient one when the balance is set to zero. In the following section we show that the balance condition is both necessary and sufficient for the case of two CKCs.

5.1 The case of two CKCs

In this section, we assume there are only two CKCs. This assumption simplifies the analysis, as the case of two CKCs can be resolved using our prior results, allowing us to examine all possible loops directly. Formally, Proposition 3 states that, given two CKCs, the necessary condition of an F_2-balanced loop from Theorem 2 is also a sufficient condition.

To build intuition, consider the scenario with two CKCs depicted in Figure 11, featuring an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$ across four states. Fix some τ_2 and assume the loop is F_2-balanced. There are then only two possibilities: either the loop is F_2-non-informative, as shown in cases (a) and (b) in Figure 11, or it is also an F_2-loop, illustrated in case (c) in Figure 11. The first possibility was covered in Theorem 3, while the second allows Oracle 1 to meet the constraints imposed by the F_1-loop when attempting to mimic τ_2.

Proposition 3. *Assume there are only two* CKCs*. Then,* Oracle 1 *dominates* Oracle 2 *if and only if F_1 refines F_2 in every* CKC *and any F_1-loop is F_2-balanced.*

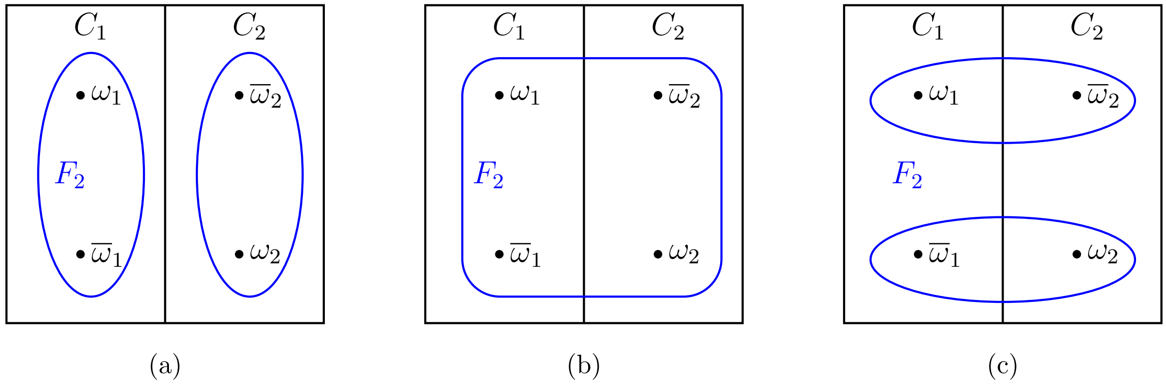


Figure 11: Two CKCs with an F_1-loop described by $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2)$. Graph (a) and (b) depict two F_2-balanced loops, that are also F_2-non-informative, and (c) describes an F_2-loop. Any other structure of F_2 yields a non-balanced loop.

6 Equivalent oracles

In this section we tackle a parallel question to dominance, which is the problem of oracles' equivalence. Specifically, we characterize necessary and sufficient conditions such that both oracles dominate one another simultaneously, as formally given in the following definition:

Definition 7. *F_1 is* equivalent *to F_2, denoted $F_1 \sim F_2$, if the two oracles dominate one another, that is, if $F_i \succeq_{\mathrm{NE}} F_{-i}$ for every $i = 1, 2$.*

Based on the results for the case that loops do not exist and the case of two CKCs, equivalence between oracles obviously requires two-sided refinement within every CKC (i.e., equivalence), and that every F_i-loop is F_{-i}-balanced for every Oracle i. This, however, is insufficient and equivalence also requires that every irreducible F_i-loop with at least 6 states is also an irreducible F_{-i}-loop. This result is given in the following Theorem 4.

Theorem 4. *F_1 is* equivalent *to F_2 if and only if for every* Oracle *i, the partition F_i refines F_{-i} in every* CKC*, any F_i-loop has a cover of F_{-i}-loops, and every irreducible F_i-loop with at least* 6 *states is an irreducible F_{-i}-loop.*

The equivalence condition concerning irreducible loops is based on the ability of both oracles to follow similar measurability constraints when signaling to players in every CKC. That is, if one oracle is constrained by an information loop, then we require the other to follow suit. Yet, this still raises the question of why we need to focus on irreducible loops. To understand this, consider a single partition element of F_i that intersects at least two CKCs where each intersection contains at least two states. This evidently generates a non-informative loop, because all pairs are non-informative. But as long as the other oracle cannot distinguish between the two states in each pair, the ability to separate different pairs in different CKCs is not needed, as each pair is common knowledge among the players themselves within every CKC.

The proof of Theorem 4 also builds on an intermediate irreducibility notion that we refer to as *type-2 irreducible loop*. More formally, an F_i-loop is type-2 irreducible if it does not have four states from the same partition element of F_i. This notion refines that of fully-informative loops (as every type-2 irreducible loop is fully-informative), but also weakens that of irreducible loops, because a type-2 irreducible loop can intersect the same CKC multiple times, and so be decomposed into sub-loops.

The notion of type-2 irreducible loops is crucial for our analysis and results, but also in a more general manner. We use type-2 irreducible loops to generate the basic elements, *building blocks*, upon which two oracles must match one another (in terms of their information). These building blocks are referred to as *clusters* and they are constructed as follows. First, we take the set of type-2 irreducible loops. Then, we consider such loops that intersect the same CKC and consider them as connected. Next, we take the transitive-closure of this relation, which yield disjoint sets of connected type-2 irreducible loops. Finally, we take every such set (of connected loops) and consider all the CKCs that it intersects - this is a cluster. We prove that the oracles' partitions match one another in each of these clusters. That is, the clusters are the basic structure upon which we derive an equivalence, and later extend it to "simpler" connections between clusters that involve only a single partition element of F_i.

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A Appendices

A.1 Key results from the companion Part I

A.1.1 Proportionality lemma from Part I

Fix two distinct signals $\{s_1, s_2\}$ and assume that the partition $F_2 = \{A_1, A_2, \ldots, A_m\}$ has m elements, as noted. Let $p_1, p_2, \ldots, p_m$ be m distinct probabilities such that all ratios of two distinct numbers from the set $\mathbb{A} = \{p_j, 1 - p_j : j = 1, 2, \ldots, m\}$ are pairwise different.[11] Define the signaling function τ_2 such that

$$\tau_2(s_1|A_j) = 1 - \tau_2(s_2|A_j) = p_j, \quad \forall \leq j \leq m. \tag{2}$$

Given this signaling function and assuming that the state space comprises a unique CKC, Lemma 1 (from Part I) states that the condition $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$ implies that τ_1 is partially proportional to τ_2, restricted to a subset of feasible signals.

[11] To achieve this, one can consider m distinct prime numbers $r_1 < r_2 < \cdots < r_m$. Define $\mathbb{T}_0 = \mathbb{Q}$, and for every $j \geq 1$, let $\mathbb{T}_j$ be the extended field of $\mathbb{T}_{j-1}$ with $\sqrt{r_j}$. Take $p_j \in \mathbb{T}_j \setminus \mathbb{T}_{j-1}$.

Lemma 1. *Fix* τ_2 *given in* Equation (2) *and a unique* CKC. *If* $\mathrm{Post}(\tau_1) \subseteq \mathrm{Post}(\tau_2)$*, then for every signal* $t \in \mathrm{Supp}(\tau_1)$ *there exists a signal* $s \in \{s_1, s_2\}$ *and a constant* $c > 0$ *such that* $\tau_1(t|\omega) = c\tau_2(s|\omega)$ *for every* $\omega \in \Omega$*.*

A.1.2 Unique CKC, characterization result from Part I

Theorem 5. *Assume that* Ω *comprises a unique common knowledge component. Then, the following are equivalent:*

- F_1 *refines* F_2*;*
- $F_1 \succeq_{\mathrm{NE}} F_2$*;*
- *For every* τ_2*, there exists* τ_1*, so that* $\mathrm{Post}(\tau_1) \subseteq \mathrm{Post}(\tau_2)$*;*
- *For every* τ_2*, there exists* τ_1*, so that* $\mathrm{Post}(\tau_1) = \mathrm{Post}(\tau_2)$*;*
- *For every* τ_2*, there exists* τ_1*, so that* $\mu_{\tau_1} = \mu_{\tau_2}$*.*

A.2 Proof of Theorem 1

Proof. One direction is straightforward. Assume, to the contrary, that Oracle 1 dominates Oracle 2, but F_1 does not refine F_2 in some CKC. Denote this CKC by C_1, and consider the set of all games in which the payoffs of all players are zero in every $\omega \notin C_1$, independent of their actions. Thus, Oracle 1 dominates Oracle 2 in every game restricted to C_1, although F_1 does not refine F_2 in C_1. This contradicts the ket result from Part I (see Theorem 5 in Appendix A.1.2).

Moving on to the second part, assume to the contrary that F_1 refines F_2 in every CKC, but Oracle 1 does not dominate Oracle 2. Therefore, there exists a strategy τ_2 such that Oracle 1 cannot produce the same distribution over posteriors as τ_2. [12] The proof now splits to 4 steps.

[12] Observe that the condition that Oracle 1 can generate the same distribution over posterior profiles as Oracle 2 implies that Oracle 1 dominates Oracle 2. To see this, consider any game and any signaling strategy τ. Since the players' strategies depend on the profile of posteriors, we can then abstract away from the underlying private and public information and assume that the players play a Bayesian both Oracles can generate distributions over the profiles of posteriors, which can be generated by both Oracles.

Step 1: Mimicking sub-strategies.

We start by defining the notion of a sub-strategy, which resembles a strategy, but with induced probabilities that may sum to less than 1. Formally, a *partial distribution* $\tilde{p}$ is a non-negative function from a finite subset of S to $[0,1]$ such that $\sum_{s\in S}\tilde{p}(s) \leq 1$. A partial distribution differs from a distribution as the probabilities need not sum to 1. Let $\tilde{\Delta}(S)$ be the set of partial distributions on S, and define a *sub-strategy* $\underline{\tau} : \Omega \to \tilde{\Delta}(S)$ as an F_1-measurable function from Ω to the set of partial distributions on S. That is, $\underline{\tau}(s|\omega) \geq 0$ and $\sum_s \underline{\tau}(s|\omega) \leq 1$, for every ω and s. Evidently, every F_1-measurable strategy is a sub-strategy.

For every sub-strategy $\underline{\tau}$ and every $p \in (\Delta(\Omega))^n$, let $\mathbf{P}_{\underline{\tau}}(p)$ be the probability that $\underline{\tau}$ yields the posterior p, i.e.,

$$\mathbf{P}_{\underline{\tau}}(p) = \sum_{\substack{(\omega,s):\ \underline{\tau}(s|\omega)>0,\\ \text{and } (\mu^i_{\underline{\tau}|\omega,s})_{i\in N}=p}} \mu(\omega)\underline{\tau}(s|\omega). \tag{3}$$

Similarly, define $\mathbf{P}_{\tau_2}(p)$ for every posterior p given the stated strategy τ_2. We say that a sub-strategy $\underline{\tau}$ *mimics* τ_2 if

$$\mathbf{P}_{\underline{\tau}}(p) \leq \mathbf{P}_{\tau_2}(p), \text{ for every } p \in (\Delta(\Omega))^n. \tag{4}$$

Hence, a sub-strategy $\underline{\tau}$ mimics τ_2 if, for every posterior p, the probability that $\underline{\tau}$ generates p does not exceed the probability that τ_2 generates it. Note that the null sub-strategy (i.e., $\underline{\tau}(s|\omega) = 0$ for every ω and s) also mimics τ_2.

Consider any sub-strategy $\underline{\tau}$ that mimics τ_2. Because τ_2 generates a finite set $\text{Post}(\tau_2)$ of possible posteriors, there exists a finite number of combinations of posteriors (which does not exceed $2^{|\text{Post}(\tau_2)|}$) that every signal of $\underline{\tau}$ supports. So, if some sub-strategy uses more than $2^{|\text{Post}(\tau_2)|}$ signals, we can apply the pigeonhole principle to deduce that the additional signals support similar combinations of posteriors as some other signals. Therefore, for every such additional signal s, there exists another signal s' and a constant $c > 0$ such that $\underline{\tau}(s|\omega) = c\underline{\tau}(s'|\omega)$ for every ω, and we can unify the two signals into one. We can thus assume that there exists a finite set of signals $\underline{S}$, such that every mimicking sub-strategy (i.e., that mimics τ_2) uses only signals from $\underline{S}$.

Step 2: Optimal sub-strategies.

Let $A_{\underline{\tau}}$ be the set of sub-strategies that mimic τ_2. Note that the set of sub-strategies supported on $\underline{S}$ is compact, and the (inequality) mimicking condition, $\mathbf{P}_{\underline{\tau}}(p) \leq \mathbf{P}_{\tau_2}(p)$ for every $p \in (\Delta(\Omega))^n$, remains valid when considering a converging sequence of sub-strategies. Thus, $A_{\underline{\tau}}$ is also compact.

Consider the function $H(\underline{\tau}) = \sum_{p \in \text{Post}(\tau_2)} \mathbf{P}_{\underline{\tau}}(p)$ defined from $A_{\underline{\tau}}$ to $[0, 1]$. As a piece-wise linear function of τ, it is a continuous, so $\underline{\tau}_{1.0} = \text{argmax}_{\underline{\tau} \in \text{A}_{\underline{\tau}}} H(\underline{\tau})$ is well-defined. If $H(\underline{\tau}_{1.0}) = 1$, then $\underline{\tau}_{1.0}$ is an F_1-measurable strategy that mimics τ_2. This contradicts the original premise (that Oracle 1 cannot induce the same distribution over posteriors as τ_2), so assume to the contrary that $\underline{\tau}_{1.0}$ is a proper sub-strategy and $H(\underline{\tau}_{1.0}) < 1$. If that is the case (i.e., if $H(\underline{\tau}_{1.0}) < 1$), there exists a posterior $p^* \in \text{Post}(\tau_2)$ so that $\mathbf{P}_{\underline{\tau}_{1.0}}(p^*) < \mathbf{P}_{\tau_2}(p^*)$.

Step 3: Partially supported and connected posteriors.

For every posterior $p \in \text{Post}(\tau_2)$, let $A_p = \{\omega \in \Omega : \ p^i(\omega) > 0 \text{ for some player } i\}$ be the set of states on which p is strictly positive, contained in some CKC denoted C_p. We say that a posterior $p \in \text{Post}(\tau_2)$ is *partially supported* (PS) if $\mathbf{P}_{\underline{\tau}_{1.0}}(p) < \mathbf{P}_{\tau_2}(p)$, otherwise we say that p is *fully supported* (FS). Let us now prove a few supporting claims related to PS posteriors.

Claim 1: If p is PS, then $\sum_s \underline{\tau}_{1.0}(s|\omega) < 1$ for every state $\omega \in A_p$.

Proof. Fix a posterior p and a state ω_0 such that $(\mu^i_{\tau|\omega_0,s})_{i \in N} = p$ for some signal s and $\tau \in \{\underline{\tau}_{1.0}, \tau_2\}$. There exists a constant α_{p,ω_0}, independent of s and τ, such that $\alpha_{p,\omega_0}\mu(\omega_0)\tau(s|\omega_0) = \sum_{\omega \in A_p \setminus \{\omega_0\}} \mu(\omega)\tau(s|\omega)$. This follows from the fact that, in order to induce the posterior p, the probabilities induced by τ must maintain the same proportions along the different states in A_p, independently of either the strategy or the signal. Otherwise, the induced posterior would not match p. Thus, Equation (3) could be re-formulated as follows,

$$\begin{aligned}
\mathbf{P}_\tau(p) &= \sum_{(\omega,s):(\mu^i_{\tau|\omega,s})_{i\in N}=p} \mu(\omega)\tau(s|\omega) \\
&= \sum_{s:(\mu^i_{\tau|\omega_0,s})_{i\in N}=p} \mu(\omega_0)\tau(s|\omega_0) + \sum_{\substack{(\omega,s):\omega\in A_p\setminus\{\omega_0\},\\ \text{and } (\mu^i_{\tau|\omega,s})_{i\in N}=p}} \mu(\omega)\tau(s|\omega) \\
&= (1+\alpha_{p,\omega_0})\mu(\omega_0) \sum_{s:(\mu^i_{\tau|\omega_0,s})_{i\in N}=p} \tau(s|\omega_0),
\end{aligned}$$

which translates to

$$\sum_{s:(\mu^i_{\tau|\omega_0,s})_{i\in N}=p} \tau(s|\omega_0) = \frac{\mathbf{P}_\tau(p)}{(1+\alpha_{p,\omega_0})\mu(\omega_0)}.$$

Summing over all $p \in \text{Supp}(\tau_2)$, we get

$$\sum_s \tau(s|\omega_0) = \frac{1}{\mu(\omega_0)} \sum_{\substack{p:(\mu^i_{\tau|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_\tau(p)}{(1+\alpha_{p,\omega_0})}. \tag{5}$$

Note that the RHS holds for either $\underline{\tau}_{1.0}$ or τ_2.

Now assume, by contradiction, that p_0 is a PS posterior and $\sum_s \underline{\tau}_{1.0}(s|\omega_0) = 1$ for some state $\omega_0 \in A_{p_0}$. Using Equation (5), for both τ_2 and $\underline{\tau}_{1.0}$, we get

$$\begin{aligned}
1 &= \sum_s \tau_2(s|\omega_0) = \frac{1}{\mu(\omega_0)} \sum_{\substack{p:(\mu^i_{\tau_2|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_{\tau_2}(p)}{(1+\alpha_{p,\omega_0})} \\
1 &= \sum_s \underline{\tau}_{1.0}(s|\omega_0) = \frac{1}{\mu(\omega_0)} \sum_{\substack{p:(\mu^i_{\underline{\tau}_{1.0}|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_{\underline{\tau}_{1.0}}(p)}{(1+\alpha_{p,\omega_0})},
\end{aligned}$$

which implies that

$$\sum_{\substack{p:(\mu^i_{\tau_2|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_{\tau_2}(p)}{(1+\alpha_{p,\omega_0})} = \sum_{\substack{p:(\mu^i_{\underline{\tau}_{1.0}|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_{\underline{\tau}_{1.0}}(p)}{(1+\alpha_{p,\omega_0})} < \sum_{\substack{p:(\mu^i_{\tau_2|\omega_0,s})_{i\in N}=p,\\ \text{for some } s}} \frac{\mathbf{P}_{\tau_2}(p)}{(1+\alpha_{p,\omega_0})},$$

where the strict inequality follows from the fact that $\mathbf{P}_{\underline{\tau}_{1.0}}(p) \leq \mathbf{P}_{\tau_2}(p)$ for every posterior p, with a strict inequality for $p = p_0$. This yields a contradiction, and the result follows. $\square$

Claim 2: If $\sum_s \underline{\tau}_{1.0}(s|\omega) < 1$ for some state ω, then there exists a PS posterior p such that $\omega \in A_p$.

Proof. Assume, to the contrary, that $\sum_s \underline{\tau}_{1.0}(s|\omega_0) < 1$ for some state ω_0, and every posterior p such that $\omega_0 \in A_p$ is FS. Using Equation (5), we deduce that

$$\begin{aligned}
1 &= \sum_s \tau_2(s|\omega_0) \\
&= \frac{1}{\mu(\omega_0)} \sum_{\substack{p:(\mu^i_{\tau_2|\omega_0,s})_{i\in N}=p, \\ \text{for some } s}} \frac{\mathbf{P}_{\tau_2}(p)}{(1+\alpha_{p,\omega_0})} \\
&= \frac{1}{\mu(\omega_0)} \sum_{\substack{p:(\mu^i_{\underline{\tau}_{1.0}|\omega_0,s})_{i\in N}=p, \\ \text{for some } s}} \frac{\mathbf{P}_{\underline{\tau}_{1.0}}(p)}{(1+\alpha_{p,\omega_0})} \\
&= \sum_s \underline{\tau}_{1.0}(s|\omega_0) < 1,
\end{aligned}$$

where the first equality follows from the fact that τ_2 is a strategy, the second and fourth equations follow from Equation (5), the third equality follows from the fact that every posterior p such that $\omega_0 \in A_p$ is FP, and the last inequality is by assumption. We thus reach a contradiction, and the result follows. $\square$

We will use Claims 1 and 2 to extend $\underline{\tau}_{1.0}$, and show that it cannot be a maximum of H. For this purpose we need to define the notion of connected posteriors. Formally, we say that two posteriors $p, p' \in \text{Post}(\tau_2)$ are *connected* if there exist two states $(\omega, \omega') \in A_p \times A_{p'} \subseteq C_p \times C_{p'}$, where $C_p \neq C_{p'}$ are two distinct CKCs, such that $F_1(\omega) = F_1(\omega')$. Equivalently, in such a case, we refer to C_p and $C_{p'}$ as *connected*, as well. Let (ω, ω') and $F_1(\omega)$ be the *connection* and *connecting set* of p and p', respectively.[13] We can now relate the notion of connected posteriors to PS ones through the following claim.

Claim 3: Fix a PS posterior p and $\omega \in A_p$. Then, for every connection (ω, ω'), there exists a PS posterior p' such that $\omega' \in A_{p'} \cap F_1(\omega)$.

[13]Equivalently, we refer to (ω, ω') and $F_1(\omega)$ as the *connection* and *connecting set* of the CKCs C_p and $C_{p'}$.

Proof. Let p be a PS posterior with a connection (ω, ω') and $F_1(\omega) = F_1(\omega')$. Using Claim 1, if p is PS, then $\sum_s \underline{\tau}_{1.0}(s|\omega) < 1$ for every $\omega \in A_p$, so the F_1-measurability constraint implies that $\sum_s \underline{\tau}_{1.0}(s|\omega') < 1$. Thus, according to Claim 2, there exists a PS posterior p' such that $\omega' \in A_{p'}$, as needed. $\square$

Step 4: Extending $\underline{\tau}_{1.0}$.

Recall that p^* is a PS posterior. Let V be the set of all CKCs C_l such that there exists a sequence of PS posteriors $(p^*, p_1, \ldots, p_l)$ where every two successive posteriors are connected and $A_{p_l} \subseteq C_l$. Assume that V also contains C_{p^*}. Let $E \subseteq V^2$ be the set of couples (C, C') such that C and C' are connected, and denote by $\mathcal{P}^*$ the set of all PS connected posteriors that generate V. Clearly, (V, E) is a connected graph and we can use it to construct a sub-strategy $\underline{\tau}$ which mimics τ_2 and $\text{Post}(\underline{\tau}) = \mathcal{P}^*$. The proof proceeds by induction on the number of vertices in V.

Preliminary step: $|V| = 1$. Assume that C_{p^*} is the unique CKC in V. Because $p^* \in \text{Post}(\tau_2)$, there exists a signal s^* and state $\omega \in C_{p^*}$ such that $\tau_2(s^*|\omega) > 0$ and $(\mu^i_{\tau_2|\omega,s^*})_{i \in N} = p^*$. Define the sub-strategy $\underline{\tau}_{1.1}(s|\omega) = \tau_2(s^*|\omega)$ for every $\omega \in A_{p^*}$. Recall that F_1 refines F_2 in every CKC, therefore $\underline{\tau}_{1.1}$ is well defined. Moreover, it is a sub-strategy that mimics τ_2 and $\text{Post}(\underline{\tau}_{1.1}) = \mathcal{P}^*$, as needed.

Induction step: $|V| = m$. Assume that for every graph (V, E) where $|V| = m$, there exists a sub-strategy $\underline{\tau}_{1.m}$ that mimics τ_2, and $\text{Post}(\underline{\tau}_{1.m}) = \mathcal{P}^*$.

Induction proof for $|V| = m + 1$. Assume that $|V| = m + 1$. The distance between C_{p^*} and every vertex (i.e., every CKC) in V is defined by the shortest path between the two vertices. Denote by C_{m+1} the vertex in (V, E) with the longest path from C_{p^*}.

We argue that C_{m+1} has exactly one connecting set with the other vertices. Otherwise, assume that there are at least two connecting sets. If the two originate from the same CKC in V, then we get an F_1-loop, which cannot exist. Thus, we can assume that the two sets originate from different CKCs, denoted C and C'. Since (V, E) is a connected graph, there exists a path from C_{p^*} to each of these CKCs. Consider the two sequences of connecting sets for these two paths. If the two are pairwise disjoint, then we have an F_1-loop from C_{p^*} to C_{m+1}, which again yields a contradiction. So the sequences must coincide at some stage. Take a truncation of the sequences from the last stage in which they coincide until C_{m+1}. The origin of the two paths are connected CKCs (sharing the same connecting set), denoted C_l and C_{l+1}, so we now have two pairwise disjoint sequences between these two connected CKCs till C_{m+1}, thus generating an F_1-loop. Therefore, we conclude that there is exactly one connecting set, denoted A, between C_{m+1} and the other CKCs in V.

Consider a refinement of F_1 where A is partitioned into two disjoint sets, $A_1 = A \setminus C_{m+1}$ and $A_2 = A \cap C_{m+1}$. In such a case, $|V| = m$ and, according to the induction step, there exists a mimicking sub-strategy $\underline{\tau}_{1.m}$ supported on every PS connected posterior in $\mathcal{P}^*$ other than the ones related to the CKC C_{m+1}. Let p_{m+1} denote a PS posterior such that $A_2 \subset A_{p_{m+1}} \subseteq C_{m+1}$. In case there is more than one PS posterior, the proof works similarly because every additional posterior shares the same connecting set A.

According to the induction step, $\text{Post}(\underline{\tau}_{1.m}) = \mathcal{P}^* \setminus \{p_{m+1}\}$, so we need to extend this sub-strategy to support p_{m+1} as well. Since $p_{m+1} \in \text{Post}(\tau_2)$, there exists a signal, denoted s^* w.l.o.g., and states $\omega \in A_{p_{m+1}} \subseteq C_{m+1}$ such that $\tau_2(s^*|\omega) > 0$ and $(\mu^i_{\tau_2|\omega,s^*})_{i\in N} = p_{m+1}$. Moreover, because C_{m+1} is not connected (neither directly, nor indirectly) to the other CKCs in V under the refined F_1, we can assume that $\sum_s \underline{\tau}_{1.m}(s|A_1) > \sum_s \underline{\tau}_{1.m}(s|A_2)$. Otherwise, we can re-scale $\underline{\tau}_{1.m}$ in the different *unconnected* elements of the refined F_1. Hence, we can also assume that there exists a signal, again denoted s^* w.l.o.g., such that $\underline{\tau}_{1.m}(s^*|A_1) > 0 = \underline{\tau}_{1.m}(s^*|A_2)$.

Define the following function

$$\underline{\tau}_{1.m+1}(s|\omega) = \begin{cases} c_m \underline{\tau}_{1.m}(s|\omega), & \text{for every } (\omega, s) \text{ s.t. } \underline{\tau}_{1.m}(s|\omega) > 0, \\ c_2 \underline{\tau}_2(s^*|\omega), & \text{for every } (\omega, s) \text{ s.t. } \omega \in A_{p_{m+1}},\ s = s^*, \end{cases}$$

where the parameters $c_m > 0$ and $c_2 > 0$ are chosen to ensure that $\underline{\tau}_{1.m+1}(s^*|A_1) = \underline{\tau}_{1.m+1}(s^*|A_2)$, thus sustaining the F_1-measurability constraint across the connecting set A, and that $\underline{\tau}_{1.m+1}$ remains a sub-strategy that mimics τ_2 (ensuring that $\sum_s \underline{\tau}(s|\omega) \le 1$ for every s and ω and the that Inequality (4) holds). In conclusion, we constructed a sub-strategy that mimics τ_2 and whose support is $\mathcal{P}^*$, and this concludes the induction.

Let $\underline{\tau}_{1*}$ be the sub-strategy that mimics τ_2 and $\mathbf{P}_{\underline{\tau}_{1*}}(p) > 0$ if and only if $p \in \mathcal{P}^*$. Assume that $\underline{\tau}_{1*}$ only uses signals in some set S^*, that are not used by $\underline{\tau}_{1.0}$ (i.e., $S^* \cap \underline{S} = \phi$). Define the following sub-strategy

$$\underline{\tau}_{2.0}(s|\omega)=\begin{cases}\underline{\tau}_{1.0}(s|\omega), & \text{for every } (\omega,s) \text{ s.t. } \underline{\tau}_{1.0}(s|\omega)>0,\\ c\underline{\tau}_{1*}(s|\omega), & \text{for every } (\omega,s) \text{ s.t. } \underline{\tau}_{1*}(s|\omega)>0,\end{cases}$$

where c is a constant. Since $\underline{\tau}_{1*}(s|\omega)$ supports only PS posteriors of $\underline{\tau}_{1.0}$, for every state ω where there exists a PS posterior p of $\underline{\tau}_{1*}(s|\omega)$ such that $\omega \in A_p$, it follows from Claim 1 that $\sum_{s\in\underline{S}}\underline{\tau}_{1.0}(s|\omega)<1$. Therefore, by choosing c sufficiently small, we can ensure that $\sum_{s\in\underline{S}\cup S^*}\underline{\tau}_{2.0}(s|\omega)=\sum_{s\in\underline{S}}\underline{\tau}_{1.0}(s|\omega)+c\sum_{s\in S^*}\underline{\tau}_{1*}(s|\omega)<1$. Hence, for the extended strategy $\underline{\tau}_{2.0}(s|\omega)$, we can guarantee that for every $\omega\in\Omega$, $\sum_{s\in\underline{S}\cup S^*}\underline{\tau}_{2.0}(s|\omega)\leq 1$. We conclude that $\underline{\tau}_{2.0}$ is a sub-strategy that mimics τ_2 and $H(\underline{\tau}_{2.0})>H(\underline{\tau}_{1.0})$ due to the extension over PS posteriors. This contradicts the definition of $\underline{\tau}_{1.0}$ as a mimicking sub-strategy that maximizes H. We can thus conclude that $H(\underline{\tau}_{1.0})=1$, and $\underline{\tau}_{1.0}$ is an F_1-measurable strategy that mimics τ_2, as needed.

□

A.3 Proof of Proposition 1

Proof. **iii** $\Rightarrow$ **i**. Suppose that $(\omega_1,\overline{\omega}_1,\omega_2,\overline{\omega}_2,\ldots,\omega_m,\overline{\omega}_m)$ is not F_2-balanced. It means that there is a partition $\{A,B\}$ s.t. $\#(A\to B)\neq\#(B\to A)$. Define

$$f(\omega)=\begin{cases}1, & \text{if } \omega\in A,\\ 2, & \text{if } \omega\in B.\end{cases}$$

We obtain,

$$\prod_{i=1}^{m}\frac{f(\omega_i)}{f(\overline{\omega}_i)}=\left(\frac{1}{2}\right)^{\#(A\to B)}\cdot 2^{\#(B\to A)}\neq 1.$$

This contradicts **iii**.

i $\Rightarrow$ **ii**. Assume **i** . For every i, let $D_i=\{\omega_j;\omega_j\in F_2(\omega_i)\}\cup\{\overline{\omega}_j;\overline{\omega}_j\in F_2(\omega_i)\}$ be the set which contains all the states in the loop that share the same information set of F_2 as ω_i. Condition **i** implies that for every ω_i, the partition $A=D_i$ and $B=(D_i)^c$ satisfies $\#(A\to B)=\#(B\to A)$. Note that $|\{\omega_j;\omega_j\in F_2(\omega_i)\}|=\#(A\to B)+\#(A\to A)$, and

$|\{\overline{\omega}_j;\overline{\omega}_j \in F_2(\omega_i)\}| = \#(B \to A) + \#(A \to A)$, where $\#(A \to A) = |\{i \in \{1,...,m\};\omega_i \in A,\ \overline{\omega}_i \in A\}|$. It follows from $\#(A \to B) = \#(B \to A)$ that

$$|\{\omega_j;\omega_j \in F_2(\omega_i)\}| = |\{\overline{\omega}_j;\overline{\omega}_j \in F_2(\omega_i)\}| \tag{6}$$

for every ω_i.

Define $J = \{i;\omega_i \in F_2(\overline{\omega}_i)\}$. We show that the rest of the states are decomposed into F_2-loops. Specifically, we show that if a finite set $S = \{(\omega_j,\overline{\omega}_j);\ \overline{\omega}_j \notin F_2(\omega_j)\}$, not necessarily an F_1-loop, satisfies Eq. (6) for every $\omega_i \in S$, then it is covered by F_2-loops.

When $|S| = 2$, Eq. (6) implies that this is an F_2-loop. We now assume the induction hypothesis: if Eq. (6) is satisfied for a set $S = \{(\omega_j,\overline{\omega}_j)\}$ and for every $\omega_i \in S$, and S contains less than or equal to m pairs, then it is covered by F_2-loops. We proceed by showing this statement for sets S containing $m+1$ pairs.

We start at an arbitrary pair, say $(\omega_1,\overline{\omega}_1)$, and show that it belongs to an F_2-loop. Once this F_2-loop is formed, the states outside of this loop satisfy Eq. (6) for every ω_i outside of this loop. By the induction hypothesis, this set is covered by F_2-loops.

Due to Eq. (6), there is at least one $\overline{\omega}_j$ such that $\overline{\omega}_j \in F_2(\omega_1)$. Consider now the two pairs, $(\omega_j,\overline{\omega}_j,\omega_1,\overline{\omega}_1)$. If this is a loop, Eq. (6) remains true when applied to the states out of this loop. The induction hypothesis completes the argument. Otherwise, there is $\overline{\omega}_k$ where $k \neq 1,j$, such that $\overline{\omega}_k \in F_2(\omega_j)$. Consider now the three pairs, $(\omega_k,\overline{\omega}_k,\omega_j,\overline{\omega}_j,\omega_1,\overline{\omega}_1)$. If this is an F_2-loop, the other states satisfy Eq. (6), and as before, this set is covered by F_2-loops. However, if this is not an F_2-loop, Eq. (6) remains true, we annex another pair and continue this way until we obtain an F_2-loop. This loop might cover the entire set, but if not, the remaining states are, by the induction hypothesis, covered by F_2-loops. This shows **ii**.

ii $\Rightarrow$ **iii**. Let $f : \{\omega_1,\overline{\omega}_1,\omega_2,\overline{\omega}_2,\ldots,\omega_m,\overline{\omega}_m\} \to (0,\infty)$ be a positive and F_2-measurable function. Suppose that $I_1,...,I_r$ is a partition of $\{1,...,m\}$, and for each $t = 1,...,r$, the set $\Big((\omega_i,\overline{\omega}_i)\Big)_{i\in I_t}$ is an F_2-loop. Since, $\Big((\omega_i,\overline{\omega}_i)\Big)_{i\in I_t}$ is an F_2-loop,

$$\prod_{i\in I_t}\frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1,$$

which implies that

$$\prod_{i=1}^{m} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = \prod_{t=1}^{r} \prod_{i \in I_t} \frac{f(\omega_i)}{f(\overline{\omega}_i)} = 1.$$

This proves **iii**. □

A.4 Proof of Proposition 2

Proof. Fix an F_i-loop $L_i = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I}$ where $I = \{1, 2, \ldots, m\}$. Let C_j denote the CKC that contains every pair $(\omega_j, \overline{\omega}_j)$.

Proof for first statement: Assume that L_i intersects the same CKC at least twice, so that $C_{l_1} = C_{l_2}$, where $l_1 < l_2$, is such CKC. Because L_i is a loop, the two pairs $(\omega_{l_1}, \overline{\omega}_{l_1})$ and $(\omega_{l_2}, \overline{\omega}_{l_2})$ that are in this CKC cannot be adjacent in the loop L_i, i.e., $l_1 \neq l_2 \pm 1$. Define the following sub-loop of L_i by omitting every state from $\overline{\omega}_{l_1}$ to ω_{l_2}. Formally, $L_i' = (\omega_1, \overline{\omega}_1, \ldots, \overline{\omega}_{l_1-1}, \omega_{l_1}, \overline{\omega}_{l_2}, \omega_{l_2+1}, \ldots, \omega_m, \overline{\omega}_m)$. This is a well-defined sub-loop of L_i (as $\omega_{l_1}, \overline{\omega}_{l_2} \in C_{l_1}$ while all other parts of the sub-loop match those of L_i), which implies that L_i is not irreducible. Note that the part we truncated from the loop L_i also forms a sub-loop, namely $L_i'' = (\omega_{l_2}, \overline{\omega}_{l_1}, \omega_{l_1+1}, \overline{\omega}_{l_1+1}, \ldots, \omega_{l_2-1}, \overline{\omega}_{l_2-1})$.

Proof for second statement: Assume, by contradiction, that L_i is irreducible, yet it has a pair of states $(\omega_l, \overline{\omega}_l)$ such that $\overline{\omega}_l \in F_i(\omega_l)$. This implies that $\{\overline{\omega}_{l-1}, \omega_l, \overline{\omega}_l, \omega_{l+1}\} \subseteq F_i(\omega_l) = F_i(\omega_{l+1})$. We can assume that $C_{l-1} \neq C_{l+1}$, otherwise the first statement suggests that L_i is not irreducible. So, define the following sub-loop of L_i by $L_i' = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I \setminus \{l\}}$. Note that L_i' is a well-defined sub-loop, as $C_{l-1} \neq C_{l+1}$ and $\overline{\omega}_{l-1} \in F_i(\omega_{l+1})$, thus contradicting the irreducible property.

Proof for third statement: Assume, w.l.o.g., that $F_i(\omega_1) \neq F_i(\overline{\omega}_1)$. If L_i intersects the same CKC twice, then we can follow the proof of the first statement, truncate the loop, and take a sub-loop that has an informative pair of states and intersects every CKC at most once. Thus, w.l.o.g., assume that L_i intersects every CKC at most once. Denote the set of informative pairs by $I^c = \{j : F_i(\omega_j) \neq F_i(\overline{\omega}_j)\}$ and define the following ordered sub-loop of L_i by $L_i' = \Big((\omega_j, \overline{\omega}_j)\Big)_{j \in I^c}$. In simple terms, L_i' is generated from L_i by truncating all non-informative pairs $(\omega_j, \overline{\omega}_j)$, where $F_i(\omega_j) = F_i(\overline{\omega}_j)$, similarly to the process used in the proof of the second statement. Focusing on L_i', note that: (i) all pairs are pairwise disjoint; (ii) every CKC is crossed at most once; (iii) $\omega_{j+1} \in F_i(\overline{\omega}_j)$ as we removed only non-informative pairs; and (iv) $\omega_j \neq \overline{\omega}_j$ are both in the same CKC as in the original loop. Hence, L_i' is a well-defined loop and an F_i-fully-informative sub-loop of L_i.

Proof of fourth statement: If the loop L_i is irreducible, then the statement holds. Otherwise, it is not irreducible and we will prove by induction on the number of pairs m in L_1. If $m = 2$, then L_i is irreducible. If $m = 3$ and L_i is not irreducible, then it has a sub-loop with two pairs. Assume w.l.o.g. that this sub-loop is based on the states $\{\omega_1, \overline{\omega_1}, \omega_2, \overline{\omega}_2\}$. It cannot be that $F_i(\overline{\omega}_1) = F_i(\overline{\omega}_2)$, because that would make $(\omega_2, \overline{\omega}_2)$ a non-informative pair. So the sub-loop is $(\omega_1, \overline{\omega_1}, \omega_2, \overline{\omega}_2)$ such that $F_i(\omega_1) = F_i(\overline{\omega}_2)$, but $F_i(\omega_1) = F_i(\overline{\omega}_3)$ and $F_i(\overline{\omega}_2) = F_i(\omega_3)$, so the pair $(\omega_3, \overline{\omega}_3)$ is non-informative.

Assume the statement holds for $m = k$ pairs, and consider an L_i loop with $k+1$ pairs. If the loop intersects the same CKC more than once, we can split is to two sub-loops (as previously done), and use the induction hypothesis for each. Hence, we can assume that the loop does not intersect the same CKC twice.

Because the loop is not irreducible, there are two states ω_{i_1} and $\overline{\omega}_{i_2}$ that are not adjacent in the loop (so $i_1 \geq i_2 + 2$), yet $F_i(\omega_{i_1}) = F_i(\overline{\omega}_{i_2})$. The last equality also suggests that $F_i(\overline{\omega}_{i_1-1}) = F_i(\omega_{i_2+1})$. If $i_1 = i_2 + 2$, then there exists only one pair between the two states. This implies that the pair $(\omega_{i_2+1}, \overline{\omega}_{i_2+1}) = (\omega_{i_1-1}, \overline{\omega}_{i_1-1})$ is non-informative, contradicting the fact that L_i is F_i-fully-informative. So we conclude that $i_1 \geq i_2 + 3$. Define the following two loops $L_i' = (\omega_{i_1}, \overline{\omega}_{i_1}, \ldots, \omega_{i_2}, \overline{\omega}_{i_2})$ and $L_i'' = (\omega_{i_2+1}, \overline{\omega}_{i_2+1}, \ldots, \omega_{i_1-1}, \overline{\omega}_{i_1-1})$, where the ordering of states follows the original loop L_i. These are two well-defined F_i-loops with less than $k+1$ pairs each, so the induction hypothesis holds and the result follows.

If L_i does not intersect the same CKC more than once and does not have at least 4 states in the same partition element, then it is irreducible.

Proof of fifth statement: If the loop has a non-informative pair $\omega_j \in F_i(\overline{\omega}_i)$, then it contains 4 states from the same partition element, so assume that the loop is F_i-fully-informative and that it does not intersect the same CKC more than once. Thus, we need to prove that it has at least 4 states in the same partition element of F_i.

Consider the strict sub-loop L_i^- of L_i. It consists of pairs, taken from the original loop. Because L_i does not intersect the same CKC more than once, all the pairs of L_i^- are a strict subset of the pairs of L_i. This implies that some pairs were omitted from L_i when generating L_i^-, so assume w.l.o.g. that the pair $\{\omega_1, \overline{\omega}_1\}$ is not included in L_i^-. This implies that one pair $\{\omega_j, \overline{\omega}_j\}$ precedes in L_i^- a different one that it precedes in L_i. That is, $F_i(\overline{\omega}_j) = F_i(\omega_{j+1})$ according to L_i, whereas $F_i(\overline{\omega}_j) = F_i(\omega_k)$ where $k \neq j+1$, according to L_i^-. But also $F_i(\omega_k) = F_i(\overline{\omega}_{k-1})$ according to L_i. Thus, $\{\overline{\omega}_j, \omega_{j+1}, \omega_k, \overline{\omega}_{k-1}\}$ are in the same partition element of L_i, as stated and the result follows. $\square$

A.5 Proof of Theorem 2

Proof. Suppose that Oracle 1 dominates Oracle 2. If there exists a CKC in which F_1 does not refine F_2, Theorem 5 from Part I (see Appendix A.1.2) states that Oracle 1 does not dominate Oracle 2 in that CKC. In other words, there exists τ_2 defined on this CKC, such that for every τ_1, it follows that $\mathrm{Post}(\tau_1) \nsubseteq \mathrm{Post}(\tau_2)$. We extend the definition of τ_2 to the entire state space in an arbitrary way, and still for every τ_1, it follows that $\mathrm{Post}(\tau_1) \nsubseteq \mathrm{Post}(\tau_2)$, and we can follow the results of Part I accordingly (specifically, the game of beliefs and Proposition 3 therein).

We proceed to show that any F_1-loop is F_2-balanced, which is equivalent to the existence of a cover by loops of F_2. Suppose, to the contrary, that an F_1-loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ is not F_2-balanced. This means that there is an F_2-measurable partition $\{A, B\}$ of these states such that Eq. (1) is not satisfied. We define an F_2-measurable signaling function that obtains two signals, α and β. Over the states of the loop, let

$$\tau_2(\alpha|\omega) = \begin{cases} x, & \text{if } \omega \in A, \\ y, & \text{if } \omega \in B, \end{cases} \tag{7}$$

and $\tau_2(\beta|\omega) = 1 - \tau_2(\alpha|\omega)$. On other states, τ_2 is defined arbitrarily. The numbers $x, y \in (0,1)$ are chosen so that $\frac{\ln x - \ln y}{\ln(1-x) - \ln(1-y)}$ is irrational.

Claim 1: If $\mathrm{Post}(\tau_1) \subseteq \mathrm{Post}(\tau_2)$, then any signal of τ_1 induces the same posteriors as α does or as β does in every CKC.

Claim 2: For any signal s of τ_1 and for any i, $\frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} \in \{\frac{x}{y}, \frac{1-x}{1-y}, \frac{y}{x}, \frac{1-y}{1-x}\}$. Therefore,

$$\prod_{i=1}^{m} \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = \left(\frac{x}{y}\right)^{\ell_1} \cdot \left(\frac{1-x}{1-y}\right)^{\ell_2} \cdot \left(\frac{y}{x}\right)^{k_1} \cdot \left(\frac{1-y}{1-x}\right)^{k_2},$$

where $\ell_1 + \ell_2 = |\{i; \omega_i \in A \text{ and } \overline{\omega}_i \in B\}|$ and $k_1 + k_2 = |\{i; \omega_i \in B \text{ and } \overline{\omega}_i \in A\}|$.

Claim 3: For any signal s of τ_1, $\prod_{i=1}^{m} \frac{\tau_1(s|\omega_i)}{\tau_1(s|\overline{\omega}_i)} = 1$.

We therefore obtain $(\frac{x}{y})^{\ell_1}(\frac{1-x}{1-y})^{\ell_2}(\frac{y}{x})^{k_1}(\frac{1-y}{1-x})^{k_2} = 1$. We conclude that there are whole numbers, say $\ell = \ell_1 - k_1$ and $k = k_2 - \ell_2$ such that $(\frac{x}{y})^{\ell} = (\frac{1-x}{1-y})^{k}$. Since $\frac{\ln x - \ln y}{\ln(1-x) - \ln(1-y)} = \frac{\ln \frac{x}{y}}{\ln \frac{1-x}{1-y}}$ is irrational, $\ell = k = 0$, implying that Eq. (1) is satisfied. This is a contradiction, so every F_1-loop is F_2-balanced.

Moving on to the third part of the theorem, fix an irreducible F_1-loop L_1, and consider an irreducible cover by a unique F_2-loop L_2, i.e., L_2 covers L_1 and both are irreducible w.r.t. the relevant partition. Note that if L_2 is also order-preserving, it implies that it *matches* L_1.

Assume, by contradiction, that L_2 is not order-preserving and the two loops do not match one another. Denote $L_1 = (\omega_1, \overline{\omega}_1, \ldots, \omega_m, \overline{\omega}_m)$ and $L_2 = (\omega_1, \overline{\omega}_1, \omega_{i_2}, \overline{\omega}_{i_2}, \ldots, \omega_{i_m}, \overline{\omega}_{i_m})$. Thus, there exist indices $k > j > 1$ such that ω_k precedes ω_j in L_2. In simple terms, it implies that though L_2 consists of the same pairs as L_1, the ordering of pairs throughout the two loops differs, as suggested in Footnote 9.

Since the two loops are irreducible, it follows from Proposition 2 that they intersect every CKC at most once and that both are fully-informative. Moreover, for every state ω in every loop L_i, every set $F_i(\omega)$ contains two states from the loop L_i (otherwise, the loop is not irreducible). So, one can define an F_i-measurable function τ_i such that $\tau_i(s|\omega_l) = \tau_i(s|\overline{\omega}_{l-1}) \neq \tau_i(s|\omega_{l'})$ for every $\omega_l \neq \omega_{l'}$ in the loop.

To simplify the exposition, partition the states of L_2 into three disjoint sets: the set $A_1^2 = \{\overline{\omega}_1, \ldots, \omega_k\}$ contains all the states of L_2 from $\overline{\omega}_1$ till ω_k (following the order of L_2), $A_k^2 = \{\overline{\omega}_k, \ldots, \omega_j\}$ contains all the states of L_2 from $\overline{\omega}_k$ till ω_j, and $A_j^2 = \{\overline{\omega}_j, \ldots, \omega_1\}$ which contains all remaining states of L_2. Follow a similar process with L_1, so that $A_1^1 = \{\overline{\omega}_1, \ldots, \omega_j\}$ contains all the states of L_1 from $\overline{\omega}_1$ till ω_j (following the order of L_1), $A_j^1 = \{\overline{\omega}_j, \ldots, \omega_k\}$ contains all the states of L_1 from $\overline{\omega}_j$ till ω_k, and $A_k^1 = \{\overline{\omega}_k, \ldots, \omega_1\}$ which contains all remaining states of

L_1.

Denote by C_l the CKC of the pair $(\omega_l, \overline{\omega}_l)$. Fix two distinct signals s_1 and s_2, and define the signaling function τ_2 as follows:

$$\tau_2(s_1|\omega) = 1 - \tau_2(s_2|\omega) = \begin{cases} p_1, & \text{if } \omega \in A_1^2 = \{\overline{\omega}_1, \dots, \omega_k\}, \\ p_2, & \text{if } \omega \in A_k^2 = \{\overline{\omega}_k, \dots, \omega_j\}, \\ p_3, & \text{if } \omega \in A_j^2 = \{\overline{\omega}_j, \dots, \omega_1\}, \\ p_4, & \text{if } \omega \in \Omega \setminus \bigcup_{i=1,j,k} A_i^2, \end{cases}$$

where the probabilities $\{p_1, p_2, p_3, p_4\}$ are chosen as in the strategy defined in Equation (2). Because the loop is irreducible, intersects every CKC at most once and F_2-fully-informative, τ_2 is a well-defined F_2-measurable function.

The result of Lemma 1 from Part I (see Appendix A.1.1) holds in every CKC of the loop (though with different probabilities). So given a CKC C_l, if there exists τ_1 such that $\text{Post}(\tau_1) \subseteq \text{Post}(\tau_2)$, then for every signal $t \in \text{Supp}(\tau_1)$ there exists a signal $s \in \{s_1, s_2\}$ and a constant $c > 0$ such that $\tau_1(t|\omega) = c\tau_2(s|\omega)$ for every $\omega \in C_l$. Therefore, in every CKC C_l and for every signal t, there exists a signal s such that $\frac{\tau_2(s|\omega_l)}{\tau_2(s|\overline{\omega}_l)} = \frac{\tau_1(t|\omega_l)}{\tau_1(t|\overline{\omega}_l)}$. Fix such a strategy τ_1.

Notice that in every CKC $C_l \neq C_1, C_j, C_k$ and for every signal $s \in \{s_1, s_2\}$, we get $\tau_2(s|\omega_l) = \tau_2(s|\overline{\omega}_l)$. Thus, $\frac{\tau_1(t|\omega_l)}{\tau_1(t|\overline{\omega}_l)} = 1$ for every t and every $l \neq i, j, k$. This implies that for every feasible signal t restricted to the loop L_1,

$$\tau_1(t|\omega) = \begin{cases} a_t, & \text{if } \omega \in A_1^1 = \{\overline{\omega}_1, \dots, \omega_j\}, \\ b_t, & \text{if } \omega \in A_j^1 = \{\overline{\omega}_j, \dots, \omega_k\}, \\ c_t, & \text{if } \omega \in A_k^1 = \{\overline{\omega}_k, \dots, \omega_1\}, \end{cases}$$

where $a_t, b_t, c_t \in (0, 1]$. Evidently, the parameters $a_t, b_t,$ and c_t can vary across the feasible signals.

In addition, Lemma 1 from Part I (see Appendix A.1.1) states that in every CKC, $\tau_1(t|\omega)$

is proportional to $\tau_2(s_i|\omega)$ for some signal $s_i \in \{s_1, s_2\}$. This yields the following constraints:

$$\frac{\tau_1(t|\omega_1)}{\tau_1(t|\overline{\omega}_1)} = \frac{c_t}{a_t} = \frac{\tau_2(s_i|\omega_1)}{\tau_2(s_i|\overline{\omega}_1)} \in \left\{\frac{p_3}{p_1}, \frac{1-p_3}{1-p_1}\right\},$$

$$\frac{\tau_1(t|\omega_j)}{\tau_1(t|\overline{\omega}_j)} = \frac{a_t}{b_t} = \frac{\tau_2(s_i|\omega_j)}{\tau_2(s_i|\overline{\omega}_j)} \in \left\{\frac{p_2}{p_3}, \frac{1-p_2}{1-p_3}\right\},$$

$$\frac{\tau_1(t|\omega_k)}{\tau_1(t|\overline{\omega}_k)} = \frac{b_t}{c_t} = \frac{\tau_2(s_i|\omega_k)}{\tau_2(s_i|\overline{\omega}_k)} \in \left\{\frac{p_1}{p_2}, \frac{1-p_1}{1-p_2}\right\}.$$

Because the two loops cover one another and specifically because L_2 is F_1-covered, Proposition 1 states that $\prod_{l=1}^{m} \frac{\tau_1(t|\omega_{i_l})}{\tau_1(t|\overline{\omega}_{i_l})} = 1$, which leaves only two possibilities for the ratios $\{\frac{c_t}{a_t}, \frac{a_t}{b_t}, \frac{b_t}{c_t}\}$ above: either they equal $\{\frac{p_3}{p_1}, \frac{p_2}{p_3}, \frac{p_1}{p_2}\}$ respectively, or $\{\frac{1-p_3}{1-p_1}, \frac{1-p_2}{1-p_3}, \frac{1-p_1}{1-p_2}\}$. This follows from the uniqueness of the ratios, as stated in Lemma 1 from Part I (see Appendix A.1.1). Note that this must hold for every feasible signal t of τ_1 across the loop.

$\tau_1(t\|\omega)$	t_1	t_2
ω_1	$\lambda_1 c_1$	$\lambda_2 c_2$
$\overline{\omega}_1$	$\lambda_1 a_1$	$\lambda_2 a_2$
ω_j	$\lambda_1 a_1$	$\lambda_2 a_2$
$\overline{\omega}_j$	$\lambda_1 b_1$	$\lambda_2 b_2$
ω_k	$\lambda_1 b_1$	$\lambda_2 b_2$
$\overline{\omega}_k$	$\lambda_1 c_1$	$\lambda_2 c_2$

Figure 12: The structure of τ_1 restricted to the states $\{\omega_1, \overline{\omega}_1, \omega_j, \overline{\omega}_j, \omega_k, \overline{\omega}_k\}$, where $\frac{c_1}{a_1} = \frac{p_3}{p_1}$, $\frac{b_1}{c_1} = \frac{p_1}{p_2}$, $\frac{c_2}{a_2} = \frac{1-p_3}{1-p_1}$ and $\frac{b_2}{c_2} = \frac{1-p_1}{1-p_2}$ and $\lambda_1, \lambda_2 > 0$.

Thus, if we focus on the states $\{\omega_1, \overline{\omega}_1, \omega_j, \overline{\omega}_j, \omega_k, \overline{\omega}_k\}$ and group together all signals t with the same distribution on these states, then for some positive constants $\lambda_1, \lambda_2 > 0$ we get the strategy defined in Figure 12. Plugging in the relevant ratios yields the probabilities given in Figure 13.

Recall that the rows must sum to 1, so that τ_1 is a well-defined strategy. So, we get the

$\tau_1(t\|\omega)$	t_1	t_2
ω_1	$\lambda_1 c_1$	$\lambda_2 c_2$
$\overline{\omega}_1$	$\lambda_1 c_1 \frac{p_1}{p_3}$	$\lambda_2 c_2 \frac{1-p_1}{1-p_3}$
ω_j	$\lambda_1 c_1 \frac{p_1}{p_3}$	$\lambda_2 c_2 \frac{1-p_1}{1-p_3}$
$\overline{\omega}_j$	$\lambda_1 c_1 \frac{p_1}{p_2}$	$\lambda_2 c_2 \frac{1-p_1}{1-p_2}$

Figure 13: The structure of τ_1 restricted to the states $\{\omega_1, \overline{\omega}_1, \omega_j, \overline{\omega}_j\}$, where probabilities are presented in terms of c_1, c_2, λ_1 and λ_2.

following system of linear equations, in which $(x, y) = (\lambda_1 c_1, \lambda_2 c_2)$ and:

$$\begin{aligned} x + y &= 1, \\ \frac{p_1}{p_3} x + \frac{1-p_1}{1-p_3} y &= 1, \\ \frac{p_1}{p_2} x + \frac{1-p_1}{1-p_2} y &= 1, \end{aligned}$$

which does not have a solution since $p_1,\ p_2,\ p_3$ are required to be distinct. Thus, we conclude that the loops must sustain the same ordering of pairs, and therefore coincide as needed. This concludes the third and final part of the theorem. □

A.6 Proof of Theorem 3

Proof. We first define an auxiliary set $\overline{\Omega}$, which groups together states that are in the same partition element of F_2 within CKCs. Formally, define the set $\overline{\Omega}$ such that $\eta(\omega') \in \overline{\Omega}$ if and only if $\eta(\omega') = \{\omega \in \Omega :\ \omega, \omega' \in C_j,\ F_2(\omega) = F_2(\omega')\}$. Accordingly, define the partition $\overline{F_2}$ to be discrete in every CKC, such that $\overline{F_2}(\eta(\omega)) = \overline{F_2}(\eta(\omega'))$ if and only if $F_2(\omega) = F_2(\omega')$. Note that $\overline{F_2}$ is essentially a projection of F_2 onto $\overline{\Omega}$. In addition, $\overline{F_1}$ is defined as follows: (i) discrete in every CKC, similarly to $\overline{F_2}$; (ii) $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ if ω and ω' are not in the same CKC, and there exist $\overline{\omega} \in \eta(\omega)$ and $\overline{\omega'} \in \eta(\omega')$ such that $F_1(\overline{\omega}) = F_1(\overline{\omega'})$; and (iii) $\overline{F_1}$ forms a partition (i.e., given (i) and (ii), if two elements of $\overline{F_1}$ contain the same state $\eta(\omega)$, they are unified into one element).

We now prove that $\overline{F_1} = \overline{F_2}$ in every CKC and that there are no $\overline{F_1}$-loops. Thus, by Theorem 1, any $\overline{F_2}$-measurable strategy $\overline{\tau_2}$ (which, extended to Ω, is also F_2-measurable) can be imitated by an $\overline{F_1}$-measurable strategy $\overline{\tau_1}$.

Step 1: $\overline{F_1} = \overline{F_2}$ in every CKC.

By definition, $\overline{F_2}$ refines $\overline{F_1}$, so we need to prove that $\overline{F_1}$ also refines $\overline{F_2}$ in every CKC. Assume, by contradiction, that $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ where ω and ω' are in the same CKC, whereas $\overline{F_2}(\eta(\omega)) \neq \overline{F_2}(\eta(\omega'))$. This suggests that $F_2(\omega) \neq F_2(\omega')$, which implies that $F_1(\omega) \neq F_1(\omega')$. According to the construction of $\overline{F_1}$, we conclude that the equality $\overline{F_1}(\eta(\omega)) = \overline{F_1}(\eta(\omega'))$ followed from the partition-formation stage described in (iii) above, through at least one other CKC. Thus, there exists an F_1-loop which connects a state in $\eta(\omega)$ with a state in $\eta(\omega')$. Without loss of generality, assume these states are ω and ω'. Because every F_1-loop is F_2-non-informative, it follows that $F_2(\omega) = F_2(\omega')$, a contradiction.

Step 2: There are no $\overline{F_1}$-loops.

An $\overline{F_1}$-loop implies that an F_1-loop exists. By construction, all Ω states in every CKC are F_2-equivalent (i.e., grouped together according to F_2). Because every F_1-loop is F_2-non-informative, it implies that the loop consists of only one $\overline{\Omega}$ state in every CKC, and not two. This contradicts the definition of a loop.

Step 3: $\overline{F_1}$ can mimic $\overline{F_2}$.

Fix a strategy τ_2, and let $\overline{\tau_2}$ be the projected strategy on $\overline{\Omega}$. Because $\overline{F_1} = \overline{F_2}$ in every CKC and there are no $\overline{F_1}$-loops, there exists an $\overline{F_1}$-measurable strategy $\overline{\tau_1}$ that imitates $\overline{\tau_2}$. Therefore, one can lift $\overline{\tau_1}$ to Ω to create τ_1, whose projection onto $\overline{\Omega}$ matches $\overline{\tau_1}$. Thus, the strategy τ_1 imitates τ_2, as needed. $\square$

A.7 Proof of Proposition 3

Proof. Denote the two CKCs by C_1 and C_2. One part of the statement follows directly from Theorem 2, so assume that F_1 refines F_2 in every CKC and any F_1-loop is F_2-balanced. If there are no F_1-loops, then the result follows from Theorem 1, so assume there exists at least one F_1-loop, and every such loop is F_2-balanced.

Take any F_1-loop $(\omega_1, \overline{\omega_1}, \omega_2, \overline{\omega_2})$ with four states. We argue that either it is also an F_2-loop or it is F_2-non-informative. Otherwise, we can assume (without loss of generality) that $F_2(\omega_1) \neq F_2(\overline{\omega_i})$, for every $i = 1, 2$. So, there are only two possibilities left: either $F_2(\omega_1) = F_2(\omega_2)$ or $F_2(\omega_1) \neq F_2(\omega_2)$. If $F_2(\omega_1) = F_2(\omega_2)$, then there exists an F_2-measurable partition of the four states such that $A = \{\omega_1, \omega_2\}$ and $B = \{\overline{\omega_1}, \overline{\omega_2}\}$, which is not balanced. Otherwise, there exists another non-balanced F_2-measurable partition of the form $A = \{\omega_1\}$ and $B = \{\overline{\omega_1}, \omega_2, \overline{\omega_2}\}$. In any case, we get a contradiction.

The proof now splits into two cases: either there exists an F_1-loop $(\omega_1, \overline{\omega_1}, \omega_2, \overline{\omega_2})$ and an index i such that $F_2(\omega_i) \neq F_2(\overline{\omega_i})$, or every such loop is F_2-non-informative. If indeed every such loop is F_2-non-informative, Theorem 3 states that Oracle 1 dominates Oracle 2, so we need only focus on the former.

Assume that there exists an F_1-loop $(\omega_1, \overline{\omega_1}, \omega_2, \overline{\omega_2})$ and an index i such that $F_2(\omega_i) \neq F_2(\overline{\omega_i})$. Denote this couple by $\{\omega_1, \overline{\omega_1}\} \subseteq C_1$. The previous conclusion implies that it is also an F_2-loop. We claim that, under these conditions, every τ_2 is F_1-measurable. Note that F_1 refines F_2 in every CKC, so we need to verify that for every $(\omega, \overline{\omega}) \in C_1 \times C_2$ such that $F_1(\omega) = F_1(\overline{\omega})$, it follows that $F_2(\omega) = F_2(\overline{\omega})$.

Take $(\omega, \overline{\omega}) \in C_1 \times C_2$ such that $F_1(\omega) = F_1(\overline{\omega})$. If $\omega = \omega_1$ or $\omega = \overline{\omega_1}$, then $(\omega, \overline{\omega})$ are part of the previously stated F_2-loop, so $F_2(\omega) = F_2(\overline{\omega})$. Otherwise, we can construct two new F_1-loops $(\omega, \overline{\omega}, \omega_1, \overline{\omega_2})$ and $(\omega, \overline{\omega}, \omega_2, \overline{\omega_1})$. Because $F_2(\omega_1) \neq F_2(\overline{\omega_1})$, either $F_2(\omega) \neq F_2(\omega_1)$ or $F_2(\omega) \neq F_2(\overline{\omega_1})$. The previous conclusion again implies that $(\omega, \overline{\omega})$ are a apart of an F_2-loop, so $F_2(\omega) = F_2(\overline{\omega})$, as needed. □

A.8 Proof of Theorem 4

Proof. We start by assuming that F_1 and F_2 are equivalent. According to Theorem 2, every F_i refines F_{-i} in every CKC, and every F_i-loop is covered by F_{-i}-loops. Fix an irreducible F_i-loop with at least 6 states, denoted L_i, and consider a cover by F_{-i}-loops. There are two possibilities: either the cover constitutes a single loop, or else. If the cover contains a shorter loop, say L'_{-i}, then that loop is not F_i-covered because L_i is irreducible, and this contradicts Theorem 2. Moreover, the cover cannot have non-informative pairs where $F_{-i}(\omega_i) = F_{-i}(\overline{\omega}_i)$, because the two partitions match one another in every CKC and L_i is irreducible. So, the cover consists of a single irreducible F_{-i}-loop, and Theorem 2 states that it is order-preserving. Thus, L_i and L_{-i} coincide as stated.

Moving to the other direction, assume that F_i refines F_{-i} in every CKC, that any F_i-loop has a cover of F_{-i}-loops, and every irreducible F_i-loop with at least 6 states is an irreducible F_{-i}-loop. Let us prove that Oracle 1 dominates Oracle 2 (and the reverse dominance follows symmetrically).

We start with two simple observations. First, in case F_1 has no loops, then the statement follows from previous results, so assume F_1 has loops. Second, we say that two CKCs C_1 and C_2 are connected if there exist $\omega_1 \in C_1$ and $\omega_2 \in C_2$ such that $F_1(\omega_1) = F_1(\omega_2)$. If there exists a CKC C which is not connected to any other CKC (i.e., for every $\omega \in C$, the partition element $F_1(\omega) \subseteq C$), then Oracle 1 dominates Oracle 2 conditional on that CKC and independently of all other CKCs. Thus, without loss of generality, we can assume that all CKCs are connected, either directly or sequentially.

For this part, we will need to define the notion of *type-2 irreducible* loops, which are fully-informative loops that do not have four states in the same information set of the relevant F_i.

Definition 8. *Let L_i be an F_i-loop. We say that the loop is* type-2 irreducible *if it does not have four states in the same information set (i.e., partition element) of F_i.*

We shall use this notion of type-2 irreducible F_1-loops as building blocks upon which every F_2-measurable τ_2 is also F_1-measurable. For that purpose, we start by proving in the following Claim 1 that every type-2 irreducible F_1-loop is also an F_2-loop. Next, we will extend this measurability result to every set of type-2 irreducible F_1-loops that intersect the same CKCs, and finally extend it to all CKCs that these loops intersect. This sets of CKCs, to be later defined as *clusters*, will be the basic sets upon which every F_2-measurable strategy is also F_1-measurable.

Claim 1. *Every type-2 irreducible F_1-loop L_1 is an F_2-loop.*

Proof. If L_1 is irreducible, then it is also an irreducible F_2-loop, and the result holds. Thus assume that L_1 is not irreducible. Using the fifth result in Proposition 2, we deduce that L_1 intersects the same CKC more than once. Using the proof of the first result in Proposition 2, we can decompose L_1 into two disjoint strict sub-loops of F_1. This can be done repeatedly, so that L_1 is decomposed into sub-loops that do not intersect the same CKC more than once. This implies that every such loop is type-2 irreducible. Thus, every such sub-loop is irreducible, and so it is also an F_2-loop.

Note that the decomposition process occurs *within* every relevant CKC C and that $F_1|_C = F_2|_C$. That is, once there are two pairs of the same loop within the same CKC, we can decompose the loop into two disjoint loops by rearranging these four states. So, one can reverse the process and recompose the sub-loops of F_2 to regenerate the original loop L_1, which is now also an F_2-loop, as needed. $\square$

Once we dealt with individual type-2 irreducible loops, we move to loops that intersect the same CKC. For that purpose, we need to prove the following supporting, general Claim 2 which states that every F_i-fully-informative loop L_i can be decomposed to type-2 irreducible F_i-loops.

Claim 2. *Every F_i-fully-informative loop L_i that is not type-2 irreducible can be decomposed to type-2 irreducible F_i-loops.*

Proof. The proof is done by induction on the number of pairs m in L_i. If $m = 2$, then it is irreducible, as needed. Assume that the statement holds for $m = k$, and consider a loop with $k+1$ pairs. If it is not type-2 irreducible, then it has four different states $\{\overline{\omega}_j, \omega_{j+1}, \overline{\omega}_l, \omega_{l+1}\}$ in the same information set of F_i, where $l > j+1$ and $l+1 < j$ so that the two pairs are not adjacent in the original loop L_i (otherwise, the loop has a non-informative pair). Note that additional connection may exists, but in any case ω_{j+1} is in the same partition element as $\overline{\omega}_j$, and the same holds for $\overline{\omega}_l$ and ω_{l+1}. Consider the loops $(\omega_j, \overline{\omega}_j, \omega_{l+1}, \overline{\omega}_{l+1}, \omega_{l+2}, \overline{\omega}_{l+2}, \ldots, \omega_{j-1}, \overline{\omega}_{j-1})$ and $(\omega_l, \overline{\omega}_l, \omega_{j+1}, \overline{\omega}_{j+1}, \omega_{j+2}, \overline{\omega}_{j+2}, \ldots, \omega_{l-1}, \overline{\omega}_{l-1})$. The two sub-loops are based on the original loop, other than the first pair, see Figure 14

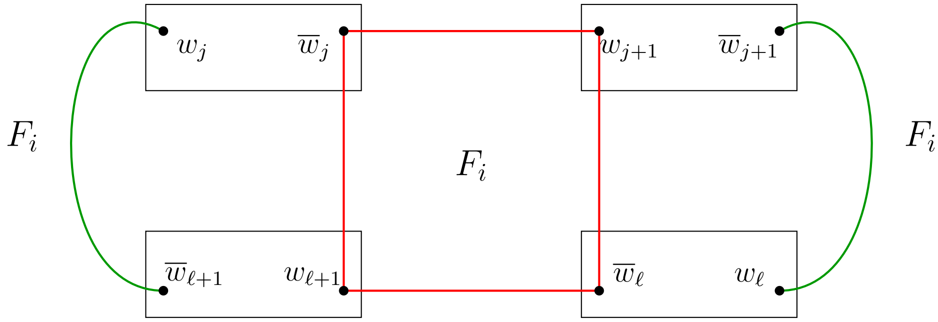

Figure 14: A fully-informative loop that is not type-2 irreducible, with four states in the same information set of F_i. The red rectangle denotes the same partition element of F_i, and the green edges denote the additional states of the original loop.

Each of these sub-loops is F_i-fully-informative, and has strictly less than k pairs. Thus, the induction hypothesis holds, and they are either type-2 irreducible, or can be separately decomposed to type-2 irreducible loops, so the result follows.

Note that even without the induction hypothesis, we can repeat the decomposition process, so that all the connections of the original loop that are based on information sets of F_i with no more than two states (in the loop) are kept in one of the sub-loops. □

Using Claim 2, we now prove in the following Claim 3, that every F_2-measurable strategy on two type-2 irreducible F_1-loops with a joint CKC (i.e., pass through the same CKC) is F_1-measurable.

Claim 3. *Fix two type-2 irreducible F_1-loops L_1 and L_1' that share at least one* CKC*. Then, every $\tau_2|_{L_1 \cup L_1'}$ is F_1-measurable.*

Proof. Fix two type-2 irreducible F_1-loop L_1 and L_1', and assume that they share at least one CKC. Denote $L_1 = (\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_m, \overline{\omega}_m)$ and $L_1' = (\omega_1', \overline{\omega}_1', \omega_2', \overline{\omega}_2', \ldots, \omega_{m'}', \overline{\omega}_{m'}')$. Assume, by contradiction, that there exists a strategy $\tau_2|_{L_1 \cup L_1'}$ which is not F_1-measurable. As already proven, each of these loops is also an F_2-loop, so the measurability constraint implies that there exist $\omega \in L_1$ and $\omega' \in L_1'$ such that $F_2(\omega) \neq F_2(\omega')$ whereas $F_1(\omega) = F_1(\omega')$. Because F_1 and F_2 match one another in every CKC, this suggests that ω and ω' are in two different CKCs. Denote a shared CKC by C_j in which there are the pairs $(\omega_j, \overline{\omega}_j)$ and $(\omega_j', \overline{\omega}_j')$ taken from L_1

and L_1' respectively. Note that the two pairs may coincide, as well as contain one of the states ω and ω', but not both (because the two are in different CKCs). See Figure 15

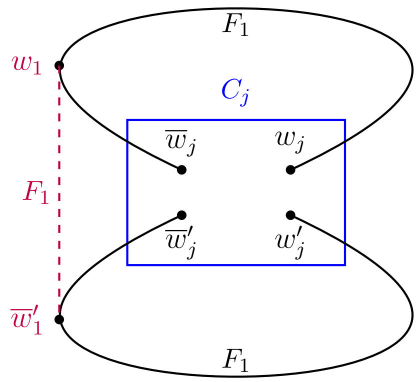

Figure 15: Two type-2 irreducible loops of F_1 that share at least one CKC.

Let us now compose a type-2 irreducible F_1 loop, using the fact that $F_1(\omega) = F_1(\omega')$. Without loss of generality, assume that $\omega = \omega_1$ and $\omega' = \overline{\omega}_1'$, and that ω_1 is not in C_j. Moreover, it cannot be the case that ω_1 and $\overline{\omega}_1'$ are both in the same loop, say L_1, because L_1 is also an F_2-loop and that would imply that either $F_2(\omega) = F_2(\omega')$ in case $\overline{\omega}_1' = \overline{\omega}_m$, or that L_1 is not a type-2 irreducible loop in case $\overline{\omega}_1' \neq \overline{\omega}_m$. Also, it must be that $F_1(\overline{\omega}_1') = F_1(\omega^*)$ where $\omega^* \in L_1$ if and only if $\omega^* \in \{\omega_1, \overline{\omega}_m\}$, otherwise L_1 is not type-2 irreducible.

We now split the proof into four possibilities:

- $\overline{\omega}_1' \in C_j$.
- $\overline{\omega}_1' \notin C_j$ and $|\{\omega_j, \overline{\omega}_j\} \cap \{\omega_j', \overline{\omega}_j'\}| = 0, 1, 2$.

Assume that $\overline{\omega}_1' \in C_j$. Consider the loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_j, \overline{\omega}_1')$. This loop matches L_1 up to state ω_j and $F_1(\omega_1) = F_1(\overline{\omega}_1')$. Thus, it is a well-defined type-2 irreducible F_1-loop, hence also an F_2-loop. Therefore, $F_2(\omega_1) = F_2(\overline{\omega}_1')$ and we reach a contradiction.

Moving on to the next possibility, assume that $\overline{\omega}_1' \notin C_j$ and $|\{\omega_j, \overline{\omega}_j\} \cap \{\omega_j', \overline{\omega}_j'\}| = 0$. Consider the loop $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \omega_j, \overline{\omega}_j', \omega_{j+1}', \overline{\omega}_{j+1}', \ldots, \omega_1', \overline{\omega}_1')$. If ω_j and $\overline{\omega}_j'$ are in different partition elements of F_1, then this is a well-defined F_1-*fully-informative* loop. If the two states are in the same partition element, then we can omit this pair from the loop and get a shorter loop (in terms of pairs). This process could be done repeatedly, until we get a well-defined F_1-*fully-informative* loop which starts with ω_1 and ends with $\overline{\omega}'_1$. If it is a type-2 irreducible F_1-loop, then it is also an F_2-loop, and $F_2(\omega_1) = F_2(\overline{\omega}'_1)$. Thus, assume that it is not type-2 irreducible, which implies that it has at least four states in the same partition element of F_1. These four states include neither ω_1 nor $\overline{\omega}'_1$, because that would imply that either L_1 or L'_1 is not type-2 irreducible. Now we can apply Claim 2, to decompose this F_1-*fully-informative* loop to type-2 irreducible F_1-loops, where at least one maintains the connection between ω_1 nor $\overline{\omega}'_1$ (see the comment at the end of the proof of Claim 2). We thus conclude that it is also an F_2-loop and $F_2(\omega_1) = F_2(\overline{\omega}'_1)$.

The next possibility is that $\overline{\omega}'_1 \notin C_j$ and $|\{\omega_j, \overline{\omega}_j\} \cap \{\omega'_j, \overline{\omega}'_j\}| = 1$. If either $\omega'_j \in \{\omega_j, \overline{\omega}_j\}$ or $\overline{\omega}'_j = \overline{\omega}_j$, then we can follow a similar proof as in the previous case where $|\{\omega_j, \overline{\omega}_j\} \cap \{\omega'_j, \overline{\omega}'_j\}| = 0$, so assume that $\overline{\omega}'_j = \omega_j$. In that case, we can redefine the previous loop by omitting ω_j and $\overline{\omega}'_j$ to get $(\omega_1, \overline{\omega}_1, \omega_2, \overline{\omega}_2, \ldots, \overline{\omega}_{j-1}, \omega'_{j+1}, \overline{\omega}'_{j+1}, \ldots, \omega'_1, \overline{\omega}'_1)$. Again, this is either a well-defined F_1-*fully-informative* loop, or could be reduced to such a loop. Applying the same arguments as before, we conclude that there exists a type-2 irreducible F_1-loop which maintains the connection between ω_1 nor $\overline{\omega}'_1$, so it is also an F_2-loop and $F_2(\omega_1) = F_2(\overline{\omega}'_1)$.

The last possibility is that $\overline{\omega}'_1 \notin C_j$ and $|\{\omega_j, \overline{\omega}_j\} \cap \{\omega'_j, \overline{\omega}'_j\}| = 2$, but in that case the analysis in the previous possibilities holds, and we reach the same conclusion that $F_2(\omega_1) = F_2(\overline{\omega}'_1)$, as needed.[14] □

Next, we extend the result of Claim 3 to more than two loops. Specifically, we say that two loops L_i and L'_i are *connected* if either they share at least one CKC, or there exists a sequence of loops starting with L_i and ending with L'_i where each two consecutive loops share at least one CKC.

Claim 4. *Consider a set A of type-2 irreducible and connected F_1-loops, i.e., every two loops are connected by one of these type-2 irreducible loops. Then, every F_2-measurable $\tau_2|_A$ is F_1-measurable.*

[14] Note that the proof of Claim 3 also holds if ω and ω' are not in the original L_1 and L'_1 loops, respectively, but are simply states in different CKCs that these loops intersect. That is, if ω and ω' are in different CKCs that L_1 and L'_1 intersect and $F_1(\omega) = F_1(\omega')$, we can construct an F_1-fully-informative loop that starts with ω and ends with ω' in a similar manner as before, and eventually conclude that $F_2(\omega) = F_2(\omega')$.

Proof. Let us prove this by induction on the number of loops. The case of two loops is proved in Claim 3, so assume the statement holds for m loops, and consider a set of $m+1$ type-2 irreducible and connected F_1-loops. Further assume, by contradiction, that there exists an F_2-measurable strategy over this set that is not F_1-measurable. Thus, there exists ω and ω' such that $F_2(\omega) \neq F_2(\omega')$ whereas $F_1(\omega) = F_1(\omega')$. Evidently, ω and ω' are in different loops and different CKCs. Denote the loops of ω and ω' by L_1 and L'_1, respectively.

If L_1 and L'_1 are connected directly (through a joint CKC) or through at most m loops (including L_1 and L'_1), then the induction hypothesis holds and every F_2-measurable strategy this set of loops is F_1-measurable, implying that $F_2(\omega) = F_2(\omega')$. Thus, assume that L_1 and L'_1 are connected through a sequence of all the $m+1$ loops (including L_1 and L_{m+1}). Note that ω' cannot be the in the same partition element as any other state from this set of loops, other than ω, the state connected to ω in L_1, and the state connected to ω' in L'_1. Otherwise, either one of these loops is not type-2 irreducible, or the F_2-measurability constraints with every intermediate loop is met (by the induction hypothesis), and again we get that $F_2(\omega) = F_2(\omega')$.

Thus, we can now follow the same stages as in the proof of Claim 3 and generate an F_1-fully-informative loop based on the sequence of loops connecting L_1 and L'_1 (as well as ω and ω'), which starts with ω_1 and ends with $\overline{\omega}'_1$. In this case, Claim 2 holds and we get a type-2 irreducible F_1-loop, which starts with ω_1 and ends with $\overline{\omega}'_1$, that is also an F_2-loop. We therefore conclude that $F_2(\omega) = F_2(\omega')$ and the induction follows accordingly. □

After we established that every F_2-measurable strategy over a set of connected loops is F_1-measurable, let us extend this result to all the CKCs that these loops intersect. For that purpose, let A be a maximal set of connected loops, where every two are connected, and let C_A be the set of all CKCs that intersect one of these loops (that is, every CKC contains a pair of states from one of these loops). We refer to every C_A as a *cluster*. We argue that every F_2-measurable strategy over a cluster C_A is F_1-measurable. To see this, recall Footnote 14 which states that the proof of Claim 3 holds for every ω and ω' in two different CKCs that intersect two connected loops L_1 and L'_1, respectively. Namely, for every two such states ω and ω' where $F_1(\omega) = F_1(\omega')$, it follows that $F_2(\omega) = F_2(\omega')$. So, as argued in the proof of Claim 4, we conclude that every F_2-measurable strategy over a cluster is F_1-measurable.

Observation 1. *Every F_2-measurable strategy over a cluster is F_1-measurable.*

Once we have established that every F_2-measurable strategy over a cluster is F_1-measurable, let us consider a partition Ω^* of Ω into clusters and individual CKCs that are not part of clusters. Note that *any* two elements of the partition Ω^* jointly intersect at most one partition element of F_1, otherwise the two components would be in the same cluster. To see this, consider the different possible intersections of elements in Ω^*. If both elements A_1 and A_2 are CKCs, then any two different partition elements of F_1 that intersect both A_1 and A_2 would form a type-2 irreducible F_1-loop. Otherwise, one of these elements is a cluster, say A_1, and it follows from previous proofs that for every ω and ω' that belong to the same cluster (but in different CKCs) and $F_1(\omega) = F_1(\omega')$, then one can form an F_1-fully-informative loop that starts with ω and ends with ω'. Thus, in case ω and ω' are in cluster A_1 and in different partition elements of F_1 that intersect A_2 (whether A_2 is a CKC or another cluster), one can form an F_1-fully-informative loop that intersects A_1 and A_2. Using Claim 2, we can conclude that A_1 and A_2 belong to the same cluster. This result is summarized in the following observation.

Observation 2. *Fix two elements $A_1, A_2 \in \Omega^*$. Then, there exists at most one partition element $F_1(\omega)$ of F_1 such that $F_1(\omega) \cap A_1$ and $F_1(\omega) \cap A_2$ are non-empty sets.*

We would now want to prove that Oracle 1 can mimic every F_2-measurable strategy defined over Ω^*. For this purpose, we present the following Lemma 2 which relates to the F_2-measurability constraints over different sets of CKCs, that are not in the same cluster (i.e., they are not connected by type-2 irreducible F_1-loops).

Lemma 2. *Fix two disjoint sets $A_1, A_2 \subseteq \Omega$ that do not intersect the same* CKCs*, and denote $A = A_1 \cup A_2$. Assume that:*

- *For every i and for every F_2-measurable $\tau_2|_{A_i}$, there exists an F_1-measurable $\tau_1^i|_{A_i}$, such that $\mu_{\tau_1}|_{A_i} = \mu_{\tau_2}|_{A_i}$.*

- *For every $\omega_1, \omega_1' \in A_1$ and $\omega_2, \omega_2' \in A_2$ such that $F_1(\omega_1) = F_1(\omega_2)$ and $F_1(\omega_1') = F_1(\omega_2')$, it follows that $F_1(\omega_1) = F_1(\omega_1')$.*

Then, for every $\tau_2|_A$, there exists $\tau_1|_A$ such that $\mu_{\tau_1}|_{A_i} = \mu_{\tau_2}|_{A_i}$ for every $i = 1, 2$.

Proof. Fix $\tau_2|_A$ and $\tau_1^i|_{A_i}$ where $i = 1, 2$, such that $\mu_{\tau_2}|_{A_i} = \mu_{\tau_1^i}|_{A_i}$ for every i. Define the sets $\tilde{A}_i = \{\omega_i \in A_i : \exists \omega_{-i} \in A_{-i}, F_1(\omega_i) = F_1(\omega_{-i})\}$ for every $i = 1, 2$. The second condition of the claim implies that all the states in $\tilde{A}_1 \cup \tilde{A}_2$ are in the same partition element of F_1. To see this, fix $\omega_1 \in \tilde{A}_1$ and, by definition, there exists a state $\omega_2 \in \tilde{A}_2$ such that $F_1(\omega_1) = F_2(\omega_2)$. If there exists another $\omega_1' \in \tilde{A}_1$, it is either connected to ω_2 (i.e., $F_1(\omega_1') = F_1(\omega_2)$), or to some $\omega_2' \in \tilde{A}_2$, and in that case the condition implies that $F_1(\omega_1) = F_1(\omega_1')$. The same holds for every $\omega_2 \in \tilde{A}_2$

For every $i = 1, 2$, let S_i be the signals induced by $\tau_1^i|_{A_i}$. Define the following strategy τ_1:

$$\tau_1((s_1, s_2)|\omega) = \begin{cases} \tau_1^1(s_1|\omega)\tau_1^2(s_2|\tilde{A}_2), & \text{if } \ \omega \in A_1, (s_1, s_2) \in S_1 \times S_2, \\ \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\omega), & \text{if } \ \omega \in A_2, (s_1, s_2) \in S_1 \times S_2. \end{cases}$$

One can easily verify that $\sum_{(s_1,s_2)} \tau_1((s_1, s_2)|\omega) = 1$ for every ω, so τ_1 is indeed a strategy.

Let us now prove that τ_1 is F_1-measurable and $\mu_{\tau_1}|_A = \mu_{\tau_2}|_A$. If we restrict τ_1 to A_i, it is clearly F_1-measurable as $\tau_1^{-i}(s_{-i}|\tilde{A}_{-i})$ is fixed for every $\omega \in A_i$ and $s_i \in S_i$. Thus, consider $\tau_1((s_1, s_2)|\omega)$ where $\omega \in \tilde{A}_1$. All the states in $\tilde{A}_1 \cup \tilde{A}_2$ are in the same partition element of F_1, so for every $(\omega_1, \omega_2) \in \tilde{A}_1 \times \tilde{A}_2$ we get

$$\begin{aligned} \tau_1((s_1, s_2)|\omega_1) &= \tau_1^1(s_1|\omega_1)\tau_1^2(s_2|\tilde{A}_2) \\ &= \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\tilde{A}_2) \\ &= \tau_1^1(s_1|\tilde{A}_1)\tau_1^2(s_2|\omega_2) \\ &= \tau_1((s_1, s_2)|\omega_2), \end{aligned}$$

and the F_1-measurability condition holds. Moreover, for every $\omega_i, \omega_i' \in A_i$ and for every (s_1, s_2) such that $\tau_1^i(s_i|\omega) > 0$ where $\omega \in \{\omega_1, \omega_1'\}$, it follows that

$$\frac{\tau_1((s_1, s_2)|\omega_i, A_i)}{\tau_1((s_1, s_2)|\omega_i', A_i)} = \frac{\tau_1^i(s_i|\omega_i)}{\tau_1^i(s_i|\omega_i')},$$

which implies that conditional on A_i, τ_1 yields the same distribution over posteriors profiles as τ_1^i, thus mimicking τ_2 on every A_i, as needed. □

We can thus finalize the proof using induction on the number of elements in Ω^*. Until now, we established in Observation 1, Observation 2 and Lemma 2 that, given either $|\Omega^*| = 1$ or $|\Omega^*| = 2$, then for every F_2-measurable strategy $\tau_2|_{\Omega^*}$, there exists $\tau_1|_{\Omega^*}$ such that $\mu_{\tau_1}|_A = \mu_{\tau_2}|_A$ for every $A \in \Omega^*$. Assume this holds for $|\Omega^*| = k \geq 2$, and consider $|\Omega^*| = k + 1$.

Denote the elements of Ω^* by $A_1, A_2, \ldots, A_k, A_{k+1}$. If there exists only one partition element of F_1 that intersects A_{k+1} and at least one A_i for $i \leq k$, then Lemma 2 holds and the result follows. Thus, assume there are at least two different partition elements $F_1(\omega) = F_1(\omega_1)$ and $F_1(\omega') = F_1(\omega_2)$ of F_1 such that $\omega, \omega' \in A_{k+1}$ and $\omega_i \in A_i$ for every $i = 1, 2$.

The proof now splits into two parts: either A_1 and A_2 are connected (i.e., there exists a sequence of partition elements of F_1 that sequentially intersect elements in $\Omega^* \setminus A_{k+1}$, starting with A_1 and ending with A_2) or A_1 and A_2 are unconnected. If they are unconnected, we can apply Lemma 2 for A_1 and A_{k+1} and then use the induction hypothesis, so we assume they are connected.

Whether A_{k+1} is a CKC or a cluster and assuming that A_1 and A_2 are connected, we argue that there exists a type-2 irreducible F_1-loop that include ω and ω', implying that A_{k+1} is part of a cluster with other elements in Ω^*. To see this, recall whenever ω and ω' belong to the same cluster and $F_1(\omega) = F_1(\omega')$, then there exists an F_1-fully-informative loop that start with ω and ends with ω'. So consider such a sequence of states $l_{\omega \to \omega'} = (\omega, \ldots, \omega')$, which would have been an F_1-loop had $F_1(\omega) = F_1(\omega')$.

Next, fix the entire path of connections of elements in Ω^* that starts with A_1 and ends with A_2. Again, the connection between A_1 and A_2 implies that there exists a sequence of states $l_{\omega_1 \to \omega_2} = (\omega_1, \ldots, \omega_2)$ in $\Omega^* \setminus A_{k+1}$, that would have been an F_1-loop had $F_1(\omega_1) = F_1(\omega_2)$. Hence, consider the sequence of states $l = (\omega, \ldots, \omega', \omega_2 \ldots, \omega_1)$ which forms an informative F_1-loop, because $F_1(\omega) \neq F_1(\omega')$. Using Proposition 2 and Claim 2, we know that this loop has a type-2 irreducible F_1-sub-loop that contains ω and ω'. Thus, A_{k+1} is in the same cluster as other elements in Ω^*, thus contradicting the assumption that $|\Omega^*| = k + 1$. $\square$